4.3.6. (AK) (a) By the computational formula, Cov(X, Y) = E[XY] - μ_Xμ_Y. We can compute

\[ \mu_y = \left( \frac{1}{4} \right)(0) + \left( \frac{1}{4} \right)(1) + \left( \frac{1}{4} \right)(-1) = 0 \]

\[ E[XY] = \left( \frac{1}{4} \right)(0 \cdot 0) + \left( \frac{1}{4} \right)(1 \cdot (-1)) + \left( \frac{1}{4} \right)(1 \cdot 1) + \left( \frac{1}{4} \right)(2 \cdot 0) = -\frac{1}{4} + \frac{1}{4} = 0. \]

Therefore, Cov(X, Y) = 0 - 0 = 0. Consider the probability P[X = 0|Y = 0].

\[ P[X = 0|Y = 0] = \frac{P[X = 0, Y = 0]}{P[Y = 0]} = \frac{1/4}{1/2} = \frac{1}{2}. \]

However, P[X = 0] = 1/4. Therefore, P[X = 0|Y = 0] ≠ P[X = 0], hence X and Y are not independent.

(b) Since X has the uniform distribution on (-1, 1), the p.d.f. of X has the constant value 1/2. Then,

\[ E[X] = \int_{-1}^{1} \frac{1}{2} x \, dx = \frac{1}{2} \left[ \frac{1}{2} - \frac{1}{2} \right] = 0. \]

and

\[ E[XY] = E[X^2] = \int_{-1}^{1} \frac{1}{2} x^2 \, dx = \frac{1}{2} \left[ \frac{1}{6} - \frac{1}{6} \right] = 0. \]

Therefore we can compute

\[ P(X \in A, Y \in B) = P(X \in A) \quad \text{and} \quad P(Y \in B) = 0, \]

Can you find p.d.f. for Y and X?

4.3.13. Let \( X_i = 1 \) or 0 respectively if person \( i \) gets their own gift. Then \( X = \sum_{i=1}^{n} X_i \) is the number of people who pick the same gift that they brought. Since each \( X_i \) has the Bernoulli distribution with success probability 1/n,

\[ E[X_i] = \frac{1}{n}, \quad \text{Var}(X_i) = \frac{1}{n} (1 - \frac{1}{n}) = \frac{n - 1}{n^2}. \]

Also,

\[ E[X_jX_k] = 1 \cdot P[X_j = 1, X_k = 1] = \frac{1}{n} \cdot \frac{1}{n - 1}, \]

hence

\[ \text{Cov}(X_j, X_k) = \frac{1}{n} \cdot \frac{1}{n - 1} - \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2(n - 1)}. \]

Therefore we can compute

\[ E[X] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1, \]

and

\[ \text{Var}(X) = \text{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{Var}(X_i) + \sum_{j,k} \text{Cov}(X_j, X_k) \]

\[ = \sum_{i=1}^{n} \frac{n - 1}{n^2} + \sum_{j,k \neq j} \frac{1}{n^2(n - 1)} \]

\[ = \frac{n - 1}{n} + \frac{n(n - 1)}{n^2(n - 1)} = \frac{n - 1}{n} + \frac{1}{n} = 1. \]

This doesn't work for \( n = 1 \).
4.3.7. The definite integrals below were computed in Mathematica.

\[
E[X] = \int_0^1 \int_0^{1-x} x \cdot 24xy \, dy \, dx = 2/5
\]

\[
E[Y] = \int_0^1 \int_0^{1-x} y \cdot 24xy \, dy \, dx = 2/5
\]

\[
E[X^2] = \int_0^1 \int_0^{1-x} x^2 \cdot 24xy \, dy \, dx = 1/5
\]

\[
E[Y^2] = \int_0^1 \int_0^{1-x} y^2 \cdot 24xy \, dy \, dx = 1/5
\]

\[
E[XY] = \int_0^1 \int_0^{1-x} xy \cdot 24xy \, dy \, dx = 2/15.
\]

Then

\[
\text{Var}(X) = E[X^2] - (E[X])^2 = 1/25; \quad \sigma_x = \sqrt{\text{Var}(X)} = 1/5.
\]

\[
\text{Var}(Y) = E[Y^2] - (E[Y])^2 = 1/25; \quad \sigma_y = \sqrt{\text{Var}(Y)} = 1/5.
\]

The covariance and correlation are

\[
\]

\[
\rho = \frac{\text{Cov}(X,Y)}{\sigma_x \sigma_y} = \frac{-2/75}{(1/5)(1/5)} = -\frac{2}{3}.
\]

The marginal density of \(X\) is

\[
f_X(x) = \int_0^{1-x} 24xy \, dy = 12x(1-x)^2, \quad x \in [0,1].
\]

Consequently the conditional density of \(Y\) given \(X = x\) is

\[
f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{24xy}{12x(1-x)^2} = \frac{2y}{(1-x)^2}, \quad y \in [0,1-x].
\]

The conditional expectation function becomes

\[
E[Y|X = x] = \int_0^{1-x} y \cdot \frac{2y}{(1-x)^2} \, dy = \frac{2y^3}{3(1-x)^2} \bigg|_0^{1-x} = \frac{2}{3}(1-x),
\]

4.4.6. Let \(X\) be the ozone reading on one day and let \(Y\) be the reading on the next day. From the covariance matrix we have

\[
\rho = \frac{204.3}{\sqrt{421.8} \sqrt{365.9}} = .52.
\]

Therefore, given \(X = 35\), \(Y\) is normal with parameters

\[
\mu_{y|x} = \mu_y + (\rho \sigma_y/\sigma_x)(x - \mu_x) = 54.7 + (.52)\frac{\sqrt{365.9}}{\sqrt{421.8}} (35 - 51.3) = 46.8,
\]

\[
\sigma_{y|x}^2 = \sigma_y^2(1 - \rho^2) = 365.9(1 - (.52)^2) = 266.96.
\]

We can then compute

\[
.90 = P[-1.645 \leq Z \leq 1.645] = P[-1.645 \leq \frac{Y - \mu_{y|x}}{\sigma_{y|x}} \leq 1.645] = P[Y \in [\mu_{y|x} - 1.645\sigma_{y|x}, \mu_{y|x} + 1.645\sigma_{y|x}]] = P[Y \in [46.8 \pm 1.645(16.34)]]).
\]

The interval endpoints in the last line work out to about \([19.9, 73.7]\).