2.5.5. The net winnings for the company are: $0 with probability \( \frac{3}{5} \) in the case that they do not win the bid; -$2 million with probability \( \left( \frac{2}{5} \right) \left( \frac{2}{3} \right) \) if they win the bid and do not find oil; or $4 million with probability \( \left( \frac{2}{5} \right) \left( \frac{1}{3} \right) \) if they win the bid and do find oil. The expectation of the net winnings is therefore

\[
0\left( \frac{3}{5} \right) - 2\left( \frac{4}{15} \right) + 4\left( \frac{2}{15} \right) = 0.
\]

2.5.7. (SA) If \( K \) = number of kings in the hand, then

\[
E[K] = \sum_{k=0}^{4} k \left( \frac{1}{5} \right) \left( \frac{48}{5^2} \right) = \frac{5}{13}.
\]

Note that the first term does not contribute anything to the expected value. Therefore, the terms of the above sum can be evaluated explicitly for \( k = 1, 2, 3, \) and 4. The answer is 0.385. For the variance,

\[
\text{Var}(K) = \sigma^2 = E[K^2] - \mu^2 = \sum_{k=0}^{4} k^2 \left( \frac{1}{5} \right) \left( \frac{48}{5^2} \right) - \left( \frac{5}{13} \right)^2
\]

= \frac{47}{51}, \quad \frac{12}{13} \cdot \frac{12}{13}.

2.5.8. \[
\frac{d}{db} E[(X - b)^2] = -2E[X - b] = -2(E[X] - b) = 0
\]

\Rightarrow b = E[X].

Since the second derivative with respect to \( b \) is \( 2 > 0 \), we have a minimum.

2.5.12. If \( x_1 \) is the dollar amount invested in the first asset, then \( (1000 - x_1) \) will be invested in the second. Let \( R_1 \) and \( R_2 \) be the two random rates of return. \( R_1 \) has mean \( \mu_1 = .05 \) and variance \( \sigma_1^2 = .0001 \), and \( R_2 \) has mean \( \mu_2 = .03 \) and variance \( \sigma_2^2 = .00005 \). The total return to the investor is \( R = x_1R_1 + (1000 - x_1)R_2. \) By the properties of mean and variance, the quantity to be maximized is

\[
E[R] - 2\text{Var}(R) = E[x_1R_1 + (1000 - x_1)R_2] - 2\text{Var}(x_1R_1 + (1000 - x_1)R_2)
\]

= \( \mu_1 \sigma_1^2 + \mu_2 \sigma_2^2 \) = \( .05\.0001 + .03\.00005 \)

\[
= -.07 + (.0003x_1 - .00003x_1^2).
\]

Set the derivative with respect to \( x_1 \), which comes out to \( .03 - .0006x_1 \), equal to 0 to obtain the critical point \( x_1 = 300 \), which is a global maximum. So, $300 should go to stock 1, and $700 to stock 2.

2.5.16. \[
E[X] = 1 \cdot P[X = 1] + 0 \cdot P[X = 0] = 1 \cdot p = p.
\]

Since the Bernoulli random variable only takes the values 0 and 1, it equals its square, hence

\[
\text{Var}(X) = E[X^2] - (E[X])^2 = E[X] - (E[X])^2 = p - p^2 = p(1 - p).
\]

2.5.22. The marginal mass function of \( X \) is \( q_x(1) = 1/4, q_x(2) = 1/2, q_x(3) = 1/4 \). The marginal mass function of \( Y \) is \( q_y(1) = 1/3, q_y(2) = 2/3 \). Therefore,

(a) \[
E[X] = \frac{1}{4}(1) + \frac{1}{2}(2) + \frac{1}{4}(3) = 2;
\]

(b) \[
E[Y] = \frac{1}{3}(1) + \frac{2}{3}(2) = \frac{5}{3};
\]

(c) \[
E[XY] = \frac{1}{12}(1)(1) + \frac{1}{6}(2)(1) + \frac{1}{12}(3)(1) + \frac{1}{6}(1)(2) + \frac{1}{3}(2)(2) + \frac{1}{6}(3)(2) = 10/3;
\]

(d) \[
E[X + Y] = E[X] + E[Y] = 2 + \frac{5}{3} = \frac{11}{3}.
\]

For this problem

\[
E(xy) = E(x)E(y),
\]

that is NOT always true.

\[
E(x+y) = E(x) + E(y) \]

is always true.
3.1.3. (a) Since the c.d.f. is the integral of this function, there is increasingly rapid growth until the peak growth rate at about \( x = -1.5 \). At \( x = 0 \) the c.d.f. levels off, then begins to grow more and more rapidly until another peak growth rate at about \( x = 1.5 \), then its growth slows until it levels off at \( x = 2 \).

(b) The p.d.f. is the derivative of this function, so it will increase to a peak at about \( x = 2 \) and then decrease.

3.1.5. (a)

\[
P[T > 50] = \int_{50}^{\infty} 0.02 \cdot e^{-0.02t} dt = e^{-0.02(50)} = e^{-1}.
\]

(b) As in part (a),

\[
P[T > 70 | T > 20] = \frac{P[T > 70]}{P[T > 20]} = \frac{e^{-0.02(70)}}{e^{-0.02(20)}} = e^{-0.02(50)} = e^{-1}
\]

Part (b) says that the chance that the bulb lasts 50 more hours once it is 20 hours old is the same as if it was new.

3.1.10. (a)

\[
\frac{1}{2} = \int_{0}^{m} 2x \, dx = m^2 \Rightarrow m = \sqrt{2}/2.
\]

(b) (SMH) Since the density is symmetric about 0, the median should be 0. Working as in part (a),

\[
F(m) = \int_{-\infty}^{m} \frac{1}{2} \exp(-|x|) \, dx
\]

\[
= \frac{1}{2} \left[ \int_{-\infty}^{0} \exp(x) \, dx + \int_{0}^{m} \exp(-x) \, dx \right]
\]

\[
= \frac{1}{2} \left[ -\exp(x) \right]_{-\infty}^{0} + \left( -\frac{1}{2} \exp(-x) \right)_{0}^{m}
\]

\[
= \frac{1}{2} \left[ -0 + (-\frac{1}{2} \exp(-m)) + \frac{1}{2} \right]
\]

\[
= 1 - \frac{1}{2} \exp(-m)
\]

\[
\frac{1}{2} = \exp(-m)
\]

Hence \( m = 0 \).

3.1.15.

\[
P[S > t] = 1 - P[S \leq t] = 1 - P[T_1 + T_2 \leq t]
\]

\[
= 1 - \int_{0}^{t} 0.02e^{-0.02t} \left( \int_{0}^{t} 0.02e^{-0.02t_2} dt_2 \right) dt_1
\]

\[
= 1 - \int_{0}^{t} 0.02e^{-0.02t_1} \cdot (1 - e^{-0.02(t_1-t_1)}) dt_1
\]

\[
= 1 - \int_{0}^{t} 0.02e^{-0.02t_1} - 0.02e^{-0.02t_1} dt_1
\]

\[
= 1 - \left( 1 - e^{-0.02} - 0.02e^{-0.02t_1} \right)
\]

\[
= 1 - \left( 1 - e^{-0.02} - 0.02te^{-0.02t} \right)
\]

Note \( f(t) = \frac{1}{2} F_S(t) = (0.02)^2 \cdot e^{-0.02t} \) is \( f(2,0.02) \). Do you see why?
3.1.14. We construct a uniform density using 6:00 p.m. as time 0. Since the arrival times are plus or minus 10 minutes, the state space becomes a square \([-10, 10] \times [10, 10]\) whose area is 400. Therefore the constant value of the density is \(1/400\). If \(X\) and \(Y\) are the two arrival times, then the event described is \(|X - Y| \leq 5\) = \(|X - 5 - Y \leq X + 5\). As shown in the figure, we must split the integral into three parts because of the varying shape of the region.

\[
P[|X - Y| \leq 5] = \int_{-10}^{10} \int_{-10}^{10} \frac{1}{400} \, dy \, dx + \int_{-10}^{10} \int_{5}^{x+5} \frac{1}{400} \, dy \, dx + \int_{5}^{10} \int_{10}^{-5} \frac{1}{400} \, dy \, dx
\]

\[
= \frac{1}{400} \left( \int_{-10}^{-5} (x + 15) \, dx + \int_{-5}^{5} 10 \, dx + \int_{5}^{10} (15 - x) \, dx \right)
\]

\[
= \frac{1}{400} \left( \frac{x^2}{2} + 15x \left[ \frac{x}{2} + 15x \right]_{-5}^{10} \right)
\]

\[
= \frac{1}{400} \cdot 175 = \frac{7}{16}.
\]