\section*{2.3.12.}

\textbf{Exercise: Red} on first 3 has \textbf{not red} on first 3 has

\begin{align*}
P[\text{at least 3}] &= P[1\text{st 3 not red, 4th red}] + P[1\text{st 4 not red, 5th red}] + P[\text{all 5 not red}] \\
&= \left(\frac{7}{13}\right)^3 \left(\frac{6}{13}\right) + \left(\frac{7}{13}\right)^4 \left(\frac{6}{13}\right) + \left(\frac{7}{13}\right)^5 \\
&= .156.
\end{align*}

\section*{2.3.13. (SA) The p.m.f. for the geometric distribution with success parameter \(p\) is as follows: \(q(n) = (1 - p)^{n-1}p\). Therefore,}

\[ F(n) = p(1 + (1 - p) + (1 - p)^2 + \ldots + (1 - p)^{n-1}). \]

This is a geometric series with \(r = (1 - p)\) which is less than 1, whose sum is \(\frac{1}{1-r}\). Substituting \((1 - p)\) for \(r\),

\[ F(n) = p \cdot \frac{1 - (1 - p)^n}{1 - (1 - p)} = 1 - (1 - p)^n. \]

Therefore

\[ 1 - F(n + m) = (1 - p)^{n+m} \]

\[ = (1 - p)^n(1 - p)^m \]

\[ = (1 - F(n))(1 - F(m)). \]

\(1 - F(n + m)\) gives the total probability that the first \(n + m\) trials are failures. This is equal to the product of the probability that the first \(n\) trials are failures and the probability that the first \(m\) trials are failures. This equality makes it clear that failing on the first \(n\) trials does not affect failing on the next \(m\) trials in any way. The outcome of each trial is independent of the outcomes of the previous ones.

\section*{2.3.15.}

The random variable counts the number of trials until the third success, hence it has the negative binomial p.m.f.

\[ \binom{k-1}{2} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^{k-3}, \quad k = 3, 4, 5, \ldots \]

To show that this function sums to 1, let us prove as a lemma the following result about series:

\[ \sum_{k=3}^{\infty} \binom{k-1}{2} x^{k-3} = \frac{1}{(1-x)^3}. \]

The left side can be rewritten

\[ \sum_{k=3}^{\infty} \binom{k-1}{2} x^{k-3} = \frac{1}{2} \sum_{k=3}^{\infty} (k-1)(k-2)x^{k-3} \]

\[ = \frac{1}{2} \left( \sum_{k=1}^{\infty} x^{k-1} \right)^"’ \]

\[ = \frac{1}{2} \left( \sum_{l=0}^{\infty} x^l \right)^"’ \]

Therefore,

\[ \sum_{k=3}^{\infty} \binom{k-1}{2} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^{k-3} = \left(\frac{1}{4}\right)^3 \sum_{k=3}^{\infty} \binom{k-1}{2} \left(\frac{3}{4}\right)^{k-3} = \left(\frac{1}{4}\right)^3 \left(1 - 3/4\right)^3 = 1. \]
2.3.18. According to the given conditions, the probabilities that an offspring is AA, aA, and aa respectively are \(1/4\), \(1/2\), and \(1/4\). The joint distribution of the numbers of each type among the 10 is multinomial

\[
P[X_1 = x_1, X_2 = x_2, X_3 = x_3] = \binom{10}{x_1, x_2, x_3} \bigg( \frac{1}{4} \bigg)^{x_1} \bigg( \frac{1}{2} \bigg)^{x_2} \bigg( \frac{1}{4} \bigg)^{x_3},
\]

where \(x_1 + x_2 + x_3 = 10\).

By itself, the number of AA offspring is \(b(10, 1/4)\), so

\[
P[X_1 \leq 2] = \sum_{x_1=0}^{2} \binom{10}{x_1} \left( \frac{1}{4} \right)^{x_1} \left( \frac{3}{4} \right)^{10-x_1} = .5256.
\]

2.4 ± 2, 5 > \(P(X=0) = q(10, \lambda) \iff \ln. \lambda > 10 \ln q\)

\(\lambda > 693.10^{-11} \iff \ln (1-p) > p \iff p > 6.93.10^{-12}\).

or approx with Poisson and \(\lambda = 10^{-12}\).

\(5 < P(X > 1) = 1 - P(X=0) = 1 - e^{-\lambda}. \therefore \lambda = 6.93 \lambda \) and \(p > 6.93.10^{-12}\).

2.4.7. The function \(q_1\) below is the p.m.f. of the binomial distribution, and \(q_2\) is the p.m.f. of the Poisson distribution. Straightforward calculation gives the results in the table.

\[
q_1(k) = \binom{30}{k} (.05)^k (.95)^{30-k}, \quad q_2(k) = \frac{e^{-1.5}1.5^k}{k!}.
\]

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<th>2</th>
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<th>4</th>
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<td>.239</td>
<td>.127</td>
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<td>.012</td>
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<td>Poisson</td>
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<td>.335</td>
<td>.251</td>
<td>.126</td>
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</tr>
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2.4.12. (a) The following sum just represents the arithmetical average of the states.

\[
\sum_{i=1}^{3} iP[X = i] = \frac{1}{5} \cdot 1 + \frac{1}{5} \cdot 2 + \frac{1}{5} \cdot 3 + \frac{1}{5} \cdot 4 + \frac{1}{5} \cdot 5 = 3.
\]

(b) \[
\sum_{i=0}^{3} iP[X = i] = 1 \cdot \binom{3}{1} \left( \frac{1}{2} \right)^3 + 2 \cdot \binom{3}{2} \left( \frac{1}{2} \right)^3 + 3 \cdot \binom{3}{3} \left( \frac{1}{2} \right)^3 = \frac{3}{2}.
\]

This sum is half of the number of trials, which is what we would expect the average value of the distribution to be. As in (a), and in (c) to follow, it represents a weighted average of states, using the probabilities as weights.

(c) \[
\sum_{i=0}^{\infty} iP[X = i] = \sum_{i=0}^{\infty} \frac{e^{-\lambda} \lambda^i}{i!} = \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} = \lambda e^{-\lambda} e^{\lambda} = \lambda.
\]
2.4.13. Using the law of total probability and the binomial theorem,

\[ P[X + Y = m] = \sum_{x=0}^{m} P[X = x, Y = m - x] \]
\[ = \sum_{x=0}^{m} e^{-2x} \frac{e^{-3m-z}}{x!} \frac{1}{(m-z)!} \]
\[ = e^{-m} \sum_{x=0}^{m} \frac{m!}{x!(m-x)!} \frac{1}{2^x \cdot 3^{m-x}} \]
\[ = 5^m \frac{e^{-m}}{m!} \]

This is the Poisson(5) p.m.f.

2.4.15. The c.d.f. is

\[ P[T_1 \leq t] = P[N_t \geq 1] = 1 - P[N_t = 0] = 1 - e^{-\lambda t}. \]

Differentiating this gives the density of \( T_1 \), which is \( \lambda e^{-\lambda t} \), as required.

2.4.16. (a) \( P[N_5 = 8] = (4 \cdot 5)^8 e^{-4.5}/8! \approx .0013. \)
(b) \( P[N_5 - N_3 = 6] = P[N_3 = 6] = (4 \cdot 3)^6 e^{-4.3}/6! \approx .0255. \)
(c) \( P[N_5 = 8|N_2 = 2] = P[N_5 - N_2 = 6|N_2 = 2] = P[N_5 - N_2 = 6] \approx .0255. \)

(d) First, \( P[N_2 = 2] = (4 \cdot 2)^2 e^{-4.2}/2! \approx .0107. \) Therefore, by the multiplication rule,

\[ P[N_2 = 2|N_5 = 8] = \frac{P[N_2 = 2 \cap N_5 = 8]}{P[N_5 = 8]} \]
\[ = \frac{P[N_5 = 8|N_2 = 2]P[N_2 = 2]}{P[N_5 = 8]} \]
\[ = .0255 \cdot .0107 \approx .21. \]

(e) Generalizing the argument in (d),

\[ P[N_2 = k|N_5 = n] = \frac{P[N_2 = k \cap N_5 = n]}{P[N_5 = n]} \]
\[ = \frac{P[N_5 = n|N_2 = k]P[N_2 = k]}{P[N_5 = n]} \]
\[ = \frac{P[N_3 = n-k]P[N_2 = k]}{e^{-3\lambda} \frac{(3\lambda)^{n-k}}{(n-k)!} \cdot e^{-2\lambda} \frac{(2\lambda)^k}{k!}} \]
\[ = e^{-5\lambda} \frac{n!}{k!(n-k)!} \cdot \frac{3^{n-k} \cdot 2^k}{5^n} \]
\[ = \binom{n}{k} \frac{3^{n-k} \cdot 2^k}{5^n} = \binom{n}{k} \left( \frac{2}{5} \right)^k \left( \frac{3}{5} \right)^{n-k} \]