Clearly show your work and answers in the space provided. It is not necessary to do the arithmetic. Follow the Honor Code.

(10) 1. You roll 5 dice. Find the probabilities that:
   a) at least 1 is a 4; b) at least 4 are a 1.

   This is a $\{5, \frac{1}{6}\}$

   a) $P(N_5 \geq 1) = 1 - P(N_5 = 0) = 1 - \left(\frac{5}{6}\right)^5$

   b) $P(N_5 \geq 4) = P(N_5 = 4) + P(N_5 = 5) = \left(\frac{1}{6}\right)^4 \cdot \frac{5}{6} + \left(\frac{1}{6}\right)^5$

(10) 2. You draw 5 cards, without replacement. Find the probabilities that:
   a) exactly one is a heart; b) at least one is a heart.

   a) choose 5 from 52; no repeat; no order

   $M(B) = \binom{52}{5}; M(A) = \binom{13}{1} \cdot \binom{39}{4}; P(A) = \frac{\binom{13}{1} \cdot \binom{39}{4}}{\binom{52}{5}}$

   b) choose 5 from 52; no repeat; in order

   $1 - \frac{39 \cdot 38 \cdot 37 \cdot 36 \cdot 5!}{4! \cdot 51 \cdot 50 \cdot 49 \cdot 48}$

(10) 3. A family has 2 children. Find the probability that both are boys, given that:
   a) the youngest is a boy; b) at least one is a boy.

   a) $B = \{(B, B), (G, B)\}$ so $P(A/B) = \frac{1}{2}$

   b) $B = \{(B, B), (G, B), (B, G), (G, G)\}$ so $P(A/B) = \frac{1}{3}$

(10) 4. Room A contains 2 women and 5 men. Room B contains 7 women and 3 men. Find the probability that you choose a woman, if you choose:
   a) a person at random; b) a room at random and then a person from the room.

   a) $\frac{9}{17}$

   b) $P(A) = \frac{1}{2} = P(B)$

   $P(W/A) = \frac{2}{7}; P(W/B) = \frac{7}{10}$

   $P(W) = \frac{1}{2} \cdot \frac{2}{7} + \frac{1}{2} \cdot \frac{7}{10} = \frac{69}{140}$
5. 5% of a population have a disease. 92% of the people with the disease test positive and 7% of people without the disease tests positive. 

Find the probabilities that a person: a) tests positive; b) has the disease, given that he tests positive.

\[ P(D) = 0.05; \quad P(D') = 0.95; \quad P(T|D) = 0.92; \quad P(T|D') = 0.07 \]

a) \[ P(T) = P(D)P(T|D) + P(D')P(T|D') = 0.05 \cdot 0.92 + 0.95 \cdot 0.07 = 0.046 + 0.0665 = 0.1125 \]

b) \[ P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{0.046}{0.1125} = 0.408 \]

6. You buy a lottery ticket. The probability that you win $100 is 0.01 and the probability that you win $5 is 0.09. Otherwise, you lose $2. Find what you expect to win or lose on average.

\[ U = 100 \cdot 0.01 + 5 \cdot 0.09 - 2 \cdot 0.9 = 1 + 0.45 - 1.8 = -0.35 \]

7. The p.d.f. of \((X, Y)\) is \(f(x, y) = \frac{x + y}{64}, 0 \leq x, y \leq 4\). Find:

a) the marginal p.d.f. of \(X\); b) the expected value of \(X\).

a) \[ f_X(x) = \int_0^4 \frac{x + y}{64} dy = \left[ \frac{x y + y^2}{128} \right]_0^4 = \frac{x^2 + 1}{16} \text{ for } 0 \leq x \leq 4 \]

b) \[ \mu_X = \int_0^4 x \left( \frac{x^3}{16} + \frac{1}{8} \right) dx = \left[ \frac{x^4}{48} + \frac{x^2}{16} \right]_0^4 = \frac{64}{48} + 1 = \frac{7}{3} \]

8. For a Poisson process with on average 2 successes per minute, find:

a) probability that there is at least one success in the first 5 minutes; b) the c.d.f. for the time until the first success; c) the p.d.f. for the time until the first success.

a) This is Poisson(2.5) so \( P(N_5 \geq 1) = 1 - P(N_5 = 0) = 1 - e^{-10} \)

b) \( P(T_1 \leq t) = P(N_t \geq 1) = 1 - P(N_t = 0) = 1 - e^{-2.5t} \)

c) \( f_1(t) = \frac{d}{dt} F_1(t) = \frac{d}{dt} \left[ 1 - e^{-2.5t} \right] = 2.5e^{-2.5t} \)