What does a return of nearly 480,000% mean? If you invested $25 in Buffett’s company in May 1965, your shares would have been worth $6,000,000 by December 2004. *(Source: Pensions and Investments)*

Of course, investments that potentially offer outrageous returns carry great risk of losing part or all of the principal. The bottom line: Is there a way to save regularly and have an investment worth one million dollars or more by retirement? In this section, we consider such savings plans, some of which come without treatment, as well as riskier investments in stocks and bonds.

**Annuities**

The compound interest formula

\[ A = P(1 + r)^t \]

gives the future value, \( A \), after \( t \) years, when a fixed amount of money, the principal, is deposited in an account that pays an annual interest rate \( r \) (in decimal form) compounded once a year. However, money is often invested in smaller amounts at periodic intervals. For example, to save for retirement, you might decide to deposit $1000 into an Individual Retirement Account (IRA) at the end of each year you retire. An **annuity** is a sequence of equal payments made at equal times. An IRA is an example of an annuity.

The **value of an annuity** is the sum of all deposits plus all interest paid. An example illustrates how to find this value.

**Example 1**

**Determining the Value of an Annuity**

You deposit $1000 into a savings plan at the end of each year for three years, with an interest rate of 8% per year compounded annually.

**a.** Find the value of the annuity after three years.

**b.** Find the interest.

**Solution**

**a.** The value of the annuity after three years is the sum of all deposits made plus interest paid over three years.

\[
\text{Value at end of year 1} = 1000
\]

\[
\text{Value at end of year 2} = 1000(1 + 0.08) + 1000 = 2080
\]

\[
\text{Value at end of year 3} = 2080(1 + 0.08) + 1000 = 3246.40
\]

The value of the annuity at the end of three years is $3246.40.

**b.** You made three payments of $1000 each, depositing a total of $3 \times$ $1000, or $3000. Because the value of the annuity is $3246.40, the interest is $3246.40 - $3000 = $246.40.
You deposit $2000 into a savings plan at the end of each year for three years. The interest rate is 10% per year compounded annually.

a. Find the value of the annuity after three years.

b. Find the interest.

Suppose that you deposit $P$ dollars into an account at the end of each year. The account pays an annual interest rate, $r$, compounded annually. At the end of the first year, the account contains $P$ dollars. At the end of the second year, $P$ dollars is deposited again. At the time of this deposit, the first deposit has received interest earned during the second year. Thus, the value of the annuity after two years is

$$P + P(1 + r).$$

The value of the annuity after three years is

$$P + P(1 + r) + P(1 + r)^2.$$

The value of the annuity after $t$ years is

$$P + P(1 + r) + P(1 + r)^2 + P(1 + r)^3 + \cdots + P(1 + r)^{t-1}.$$

Each term in this sum is obtained by multiplying the preceding term by $(1 + r)$. Thus, the terms form a geometric sequence. Using a formula for the sum of the terms of a geometric sequence, we can obtain the following formula that gives the value of this annuity:

**VALUE OF AN ANNUITY: INTEREST COMPOUNDED ONCE A YEAR**

If $P$ is the deposit made at the end of each year for an annuity that pays an annual interest rate $r$ (in decimal form) compounded once a year, the value, $A$, of the annuity after $t$ years is

$$A = \frac{PMT}{r} [(1 + r)^t - 1].$$

**EXAMPLE 2**  DETERMINING THE VALUE OF AN ANNUITY

To save for retirement, you decide to deposit $1000 into an IRA at the end of each year for the next 30 years. If you can count on an interest rate of 10% per year compounded annually,

a. How much will you have from the IRA after 30 years?

b. Find the interest.
SOLUTION

a. The amount that you will have from the IRA is its value after 30 years:

\[ A = \frac{1}{r} \left( \frac{(1 + r)^t - 1}{(1 + 0.10)^{30}} - 1 \right) \]

Use the formula for the value of an annuity.

\[ A = \frac{1000 \left[ (1 + 0.10)^{30} - 1 \right]}{0.10} \]

The annuity involves year-end deposits. The interest rate is 10%. The number of years is 30: \( t = 30 \).

\[ \approx \frac{1000(17.4494 - 1)}{0.10} \]

Using parentheses keys on a graphing calculator, this calculation can be done in a single step.

\[ \approx \frac{1000(16.4494)}{0.10} \]

Use a calculator to find \((1.10)^{30}\):

\[ \boxed{1.1^{30} = 17.4494} \]

\[ = 164,494 \]

After 30 years, you will have approximately $164,494 from the IRA.

b. You made 30 payments of $1000 each, depositing a total of $30 \times 1000$, or $30,000. Because the value of the annuity is approximately $164,494, the interest is approximately

\[ 164,494 - 30,000, \text{ or } 134,494. \]

The interest is nearly 4 1/2 times the amount of your payments, illustrating the power of compounding.

STUDY TIP

Annuities can be categorized by when payments are made. The formula in the box describes ordinary annuities, where payments are made at the end of each period. The formula assumes the same number of yearly payments and yearly compounding periods. An annuity plan in which the payments are made at the beginning of each period is called an annuity due.

VALUE OF AN ANNUITY: INTEREST COMPOUNDED TIMES PER YEAR

If \( \text{PMT} \) is the deposit made at the end of each compounding period for an annuity that pays an annual interest rate \( r \) (in decimal form) compounded \( n \) times per year, the value, \( A \), of the annuity after \( t \) years is

\[ A = \frac{\text{PMT} \left[ \left( 1 + \frac{r}{n} \right)^{nt} - 1 \right]}{\frac{r}{n}} \]

EXAMPLE 3

DETERMINING THE VALUE OF AN ANNUITY

At age 25, to save for retirement, you decide to deposit $200 at the end of each month into an IRA that pays 7.5% compounded monthly.

a. How much will you have from the IRA when you retire at age 65?

b. Find the interest.
**SOLUTION**

a. Because you are 25, the amount that you will have from the IRA when you retire at 65 is its value after 40 years.

\[
PMT \left[ \frac{1 + r}{n} \right]^n - 1\]

\[
A = \frac{r}{n} \left[ 200 \left( 1 + \frac{0.075}{12} \right)^{12\cdot40} - 1 \right]
\]

\[
= \frac{0.075}{12} \left[ 200 \left( 1.00625 \right)^{480} - 1 \right]
\]

\[
\approx \frac{0.075}{12} \left[ 0.00625 \right]
\]

\[
\approx 604,765
\]

After 40 years, you will have approximately $604,765 when retiring at age 65.

b. Interest = Value of the IRA − Total deposits

\[
\approx \$604,765 - (\$200 \cdot 12 \cdot 40)
\]

\[
= \$604,765 - \$96,000 = \$508,765
\]

The interest is approximately $508,765, more than five times the amount of your contributions to the IRA.

---

**Planning for the Future with an Annuity**

By solving the annuity formula for \( \text{PMT} \), we can determine the amount of money that should be deposited at the end of each compounding period so that an annuity has a future value of \( A \) dollars. The following formula gives the regular payments, \( \text{PMT} \), needed to reach a financial goal, \( A \):

\[
\text{PMT} = \frac{A \left( \frac{r}{n} \right)}{\left( 1 + \frac{r}{n} \right)^n - 1}
\]
Exercises 19 and 20 refer to the stock tables for Goodyear (the tire company) and JCPenney (the department store) given below. In each exercise, use the stock table to answer the following questions. Where necessary, round dollar amounts to the nearest cent.

a. What were the high and low prices for a share for the past 52 weeks?

b. If you owned 700 shares of this stock last year, what dividend did you receive?

c. What is the annual return for the dividends alone? How does this compare to a bank offering a 3% interest rate?

### 19. 52-Week
<table>
<thead>
<tr>
<th>High</th>
<th>Low</th>
<th>Stock</th>
<th>SYM</th>
<th>Div</th>
<th>Yld %</th>
<th>PE</th>
<th>Vol 100s</th>
<th>Hi</th>
<th>Lo</th>
<th>Close</th>
</tr>
</thead>
<tbody>
<tr>
<td>73.25</td>
<td>45.44</td>
<td>Goodyear</td>
<td>GT</td>
<td>1.20</td>
<td>2.2</td>
<td>17</td>
<td>5915</td>
<td>56.38</td>
<td>54.38</td>
<td>55.50</td>
</tr>
</tbody>
</table>

### 20. 52-Week
<table>
<thead>
<tr>
<th>High</th>
<th>Low</th>
<th>Stock</th>
<th>SYM</th>
<th>Div</th>
<th>Yld %</th>
<th>PE</th>
<th>Vol 100s</th>
<th>Hi</th>
<th>Lo</th>
<th>Close</th>
</tr>
</thead>
<tbody>
<tr>
<td>78.34</td>
<td>35.38</td>
<td>Penney JCP</td>
<td>JCP</td>
<td>2.18</td>
<td>4.7</td>
<td>22</td>
<td>7473</td>
<td>48.19</td>
<td>46.63</td>
<td>46.88</td>
</tr>
</tbody>
</table>

### Practice Plus

In Exercises 21–22, round all answers to the nearest dollar.

21. Here are two ways of investing $30,000 for 20 years.

- **Lump-Sum Deposit**
  - Rate: 5% compounded annually
  - Time: 20 years

- **Periodic Deposit**
  - Rate: 5% compounded annually
  - Time: 20 years

a. After 20 years, how much more will you have from the lump-sum investment than from the annuity?

b. After 20 years, how much more interest will be earned from the lump-sum investment than from the annuity?

22. Here are two ways of investing $40,000 for 25 years.

- **Lump-Sum Deposit**
  - Rate: 6.5% compounded annually
  - Time: 25 years

- **Periodic Deposit**
  - Rate: 6.5% compounded annually
  - Time: 25 years

a. After 25 years, how much more will you have from the lump-sum investment than from the annuity?

b. After 25 years, how much more interest will be earned from the lump-sum investment than from the annuity?

In Exercises 23–24,

a. Determine the deposit at the end of each month rounded up to the nearest dollar.

b. Assume that the annuity in part (a) is a tax-deferred account belonging to a man whose gross income in 20 years is $50,000. Use Table 8.1 on page 448 to calculate his taxes first with and then without the IRA. Assume the man is single with no dependents, has no tax credits, and takes the standard deduction.

c. What percent of his gross income are the man's federal taxes with and without the IRA? Round to the nearest tenth of a percent.

<table>
<thead>
<tr>
<th>Periodic Deposit</th>
<th>Rate</th>
<th>Time</th>
<th>Financial Goal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$7? at the end of each month</td>
<td>8% compounded monthly</td>
<td>40 years</td>
<td>$1,000,000</td>
</tr>
<tr>
<td>$7? at the end of each month</td>
<td>7% compounded monthly</td>
<td>40 years</td>
<td>$650,000</td>
</tr>
</tbody>
</table>

25. Solve for $P$: 

$$A = \frac{P[(1 + r)^t - 1]}{r}.$$ 

What does the resulting formula describe?

26. Solve for $P$: 

$$A = \frac{P \left( 1 + \frac{r}{n} \right)^{nt} - 1}{\frac{r}{n}}.$$ 

What does the resulting formula describe?
Application Exercises

In Exercises 27–32, use the formula

\[ A = \frac{PMT}{r} \left[ \left( 1 + \frac{r}{n} \right)^{nt} - 1 \right] \]

or

\[ A = \frac{r}{n} \left( \left( 1 + \frac{r}{n} \right)^{nt} - 1 \right) \]

Round to the nearest dollar.

To save money for a sabbatical to earn a master’s degree, you deposit $2000 at the end of each year in an annuity that pays 7.5% compounded annually.

a. How much will you have saved at the end of five years?

b. Find the interest.

To save money for a sabbatical to earn a master’s degree, you deposit $2500 at the end of each year in an annuity that pays 6.25% compounded annually.

a. How much will you have saved at the end of five years?

b. Find the interest.

At age 25, to save for retirement, you decide to deposit $50 at the end of each month in an IRA that pays 5.5% compounded monthly.

a. How much will you have from the IRA when you retire at age 65?

b. Find the interest.

At age 25, to save for retirement, you decide to deposit $75 at the end of each month in an IRA that pays 6.5% compounded monthly.

a. How much will you have from the IRA when you retire at age 65?

b. Find the interest.

Offer scholarship funds to children of employees, a company invests $10,000 at the end of every three months in an annuity that pays 10.5% compounded quarterly.

a. How much will the company have in scholarship funds at the end of ten years?

b. Find the interest.

Offer scholarship funds to children of employees, a company invests $15,000 at the end of every three months in an annuity that pays 9% compounded quarterly.

a. How much will the company have in scholarship funds at the end of ten years?

b. Find the interest.

In Exercises 33–36, use the formula

\[ PMT = \frac{A \left( \frac{r}{n} \right)}{\left( 1 + \frac{r}{n} \right)^{nt} - 1} \]

Round to the nearest dollar.

You would like to have $3500 in four years for a special vacation following graduation by making deposits at the end of every six months in an annuity that pays 5% compounded semiannually.

a. How much should you deposit at the end of every six months?

b. How much of the $3500 comes from deposits and how much comes from interest?

34. You would like to have $4000 in four years for a special vacation following graduation by making deposits at the end of every six months in an annuity that pays 7% compounded semiannually.

a. How much should you deposit at the end of every six months?

b. How much of the $4000 comes from deposits and how much comes from interest?

35. How much should you deposit at the end of each month into an IRA that pays 8.5% compounded monthly to have $4 million when you retire in 45 years? How much of the $4 million comes from interest?

36. How much should you deposit at the end of each month into an IRA that pays 6.5% compounded monthly to have $2 million when you retire in 45 years? How much of the $2 million comes from interest?

• Writing in Mathematics

37. What is an annuity?

38. What is meant by the value of an annuity?

39. Write a problem involving the formula for regular payments needed to achieve a financial goal. The problem should be similar to Example 4 on page 472. However, the problem should be unique to your situation. Include something for which you would like to save, how much you need to save, and how long it will take to achieve your goal. Then solve the problem.

40. What is stock?

41. Describe how to find the percent ownership that a shareholder has in a company.

42. Describe the two ways that investors make money with stock.

43. What is a bond? Describe the difference between a stock and a bond.

44. Using a recent newspaper, copy the stock table for a company of your choice. Then explain the meaning of the numbers in the columns.

45. If an investor sees that the dividends for a stock have a lower annual rate than those for a no-risk bank account, should the stock be sold and the money placed in the bank account? Explain your answer.

Use the following investments to answer Exercises 46–49.

- Investment 1: 1000 shares of IBM stock
- Investment 2: A 5-year bond with a 22% interest rate issued by a small company that is testing and planning to sell delicious, nearly zero-calorie desserts
- Investment 3: A 30-year U.S. treasury bond at a fixed 7% annual rate

46. Which of these investments has the greatest risk? Explain why.

47. Which of these investments has the least risk? Explain why.

48. Which of these investments has the possibility of the greatest return? Explain why.

49. If you could be given one of these investments as a gift, which one would you choose? Explain why.
EXAMPLE 4  ACHIEVING A FINANCIAL GOAL

You would like to have $20,000 to use as a down payment for a home in making regular, end-of-month deposits in an annuity that pays 8% annually.

a. How much should you deposit each month?

b. How much of the $20,000 down payment comes from deposits and how much comes from interest?

SOLUTION

\[ P = \frac{A \left( \frac{r}{n} \right)}{\left[ \left( 1 + \frac{r}{n} \right)^{nt} - 1 \right]} \]

Use the formula for regular payments, \( P \), to achieve a financial goal, \( A \).

\[ P = \frac{20,000 \left( \frac{0.08}{12} \right)}{\left[ \left( 1 + \frac{0.08}{12} \right)^{12 \times 5} - 1 \right]} \]

\[ \approx 273 \]

Your goal is to accumulate $20,000 (\( A \)) over five years (\( t = 5 \)). The interest rate (\( r = 0.08 \)) compounded monthly (\( n = 12 \)).

Use a calculator and round up to the nearest whole number to be certain you do not fall short of your goal.

You should deposit $273 each month to be certain of having $20,000 for your home payment on a home.

b. Total deposits = $273 \times 12 \times 5 = $16,380

\[ \text{Interest} = 20,000 - 16,380 = 3620 \]

We see that $16,380 of the $20,000 comes from your deposits and the remaining $3620 comes from interest.

Parents of a baby girl are in a financial position to begin saving for her college education. They plan to have $100,000 in a college fund in 18 years, making regular, end-of-month deposits in an annuity that pays 9% compounded monthly.

a. How much should they deposit each month? Round up to the nearest whole number.

b. How much of the $100,000 college fund comes from deposits and how much comes from interest?

---

Section 8.4

Check Point Exercises

1. a. $6620  b. $620  2. a. $777,170  b. $657,170  c. $333,946  d. $291,946  e. $187  f. $40,392  g. $2160  h. $42,375  i. 1.5%; This is much lower than a bank account paying 3.5%.  j. $7,203,200

Interest = $20,000 - $16,380 = $3620

Exercise Set 8.4

1. a. $66,132  b. $26,132  c. $89,334  d. $29,334  e. $702,528  f. $542,528  g. $187  h. $40,392  i. $2160  j. $42,375  k. 1.5%; This is much lower than a bank account paying 3.5%.  l. $7,203,200

The closing price is up $0.03 from the previous day's closing price.  m. $1.34
Practice Exercises
Exercises 1–10,

a. Use the formula

\[ \text{PMT} = \frac{A}{r} \left( 1 + \frac{r}{n} \right)^n - 1 \]

or

\[ \text{PMT} = \frac{A}{r} \left( 1 + \frac{r}{n} \right)^n - 1 \]

to find the value of each annuity. Round to the nearest dollar.

Find the interest.

<table>
<thead>
<tr>
<th>Periodic Deposit</th>
<th>Rate</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2000 at the end of each year</td>
<td>5% compounded annually</td>
<td>20 years</td>
</tr>
<tr>
<td>$3000 at the end of each year</td>
<td>4% compounded annually</td>
<td>20 years</td>
</tr>
<tr>
<td>$4000 at the end of each year</td>
<td>6.5% compounded annually</td>
<td>40 years</td>
</tr>
<tr>
<td>$5000 at the end of each year</td>
<td>5.5% compounded annually</td>
<td>40 years</td>
</tr>
<tr>
<td>$6000 at the end of each month</td>
<td>6% compounded monthly</td>
<td>30 years</td>
</tr>
<tr>
<td>$7000 at the end of each month</td>
<td>5% compounded monthly</td>
<td>30 years</td>
</tr>
<tr>
<td>$8000 at the end of every six months</td>
<td>4.5% compounded semiannually</td>
<td>25 years</td>
</tr>
<tr>
<td>$9000 at the end of every six months</td>
<td>6.5% compounded semiannually</td>
<td>25 years</td>
</tr>
<tr>
<td>$10000 at the end of every three months</td>
<td>6.25% compounded quarterly</td>
<td>6 years</td>
</tr>
<tr>
<td>$12000 at the end of every three months</td>
<td>3.25% compounded quarterly</td>
<td>6 years</td>
</tr>
</tbody>
</table>

Exercises 11–18,

b. Use the formula

\[ \text{PMT} = \frac{A \left( \frac{r}{n} \right)}{\left( 1 + \frac{r}{n} \right)^n - 1} \]

determine the periodic deposit. Round up to the nearest dollar.

How much of the financial goal comes from deposits and how much comes from interest?

<table>
<thead>
<tr>
<th>Periodic Deposit</th>
<th>Rate</th>
<th>Time</th>
<th>Financial Goal</th>
</tr>
</thead>
<tbody>
<tr>
<td>at the end of each year</td>
<td>6% compounded annually</td>
<td>18 years</td>
<td>$140,000</td>
</tr>
<tr>
<td>at the end of each year</td>
<td>5% compounded annually</td>
<td>18 years</td>
<td>$150,000</td>
</tr>
<tr>
<td>at the end of each month</td>
<td>4.5% compounded monthly</td>
<td>10 years</td>
<td>$200,000</td>
</tr>
<tr>
<td>at the end of each month</td>
<td>7.5% compounded monthly</td>
<td>10 years</td>
<td>$250,000</td>
</tr>
<tr>
<td>at the end of each month</td>
<td>7.25% compounded monthly</td>
<td>40 years</td>
<td>$1,000,000</td>
</tr>
<tr>
<td>at the end of each month</td>
<td>8.25% compounded monthly</td>
<td>40 years</td>
<td>$1,500,000</td>
</tr>
<tr>
<td>at the end of every three months</td>
<td>3.5% compounded quarterly</td>
<td>5 years</td>
<td>$20,000</td>
</tr>
<tr>
<td>at the end of every three months</td>
<td>4.5% compounded quarterly</td>
<td>5 years</td>
<td>$25,000</td>
</tr>
</tbody>
</table>