2.4. The Chain Rule

If \( y = f(u) \) is a differentiable function of \( u \) and \( u = g(x) \) is in turn a differentiable function of \( x \), then the composite function \( f(g(x)) \) is a differentiable function of \( x \) whose derivative is given by the product

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]

or, equivalently, by

\[
\frac{dy}{dx} = f'(g(x))g'(x)
\]
The Chain Rule

Example

Compute the derivative $\frac{dy}{dx}$ and simplify the answer if

$$y = u^2 - 3u + 4; \quad u = 1 - x^2$$
The Chain Rule

Example

Compute the derivative \( \frac{dy}{dx} \bigg|_{x=\frac{1}{2}} \) if

\[ y = u^2 - 2u + 2; \quad u = \frac{1}{x} \]
The Chain Rule

Sometimes when dealing with a composite function $y = f(g(x))$ it may help to think of $f$ as the “outer” function and $g$ as the “inner” function. Then the chain rule says that the derivative of $y = f(g(x))$ with respect to $x$ is given by the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

Example
Differentiate the following function and simplify the answer.

$$h(x) = \sqrt{x^6 - 3x^2}$$
The General Power Rule

For any real number $n$ and differentiable function $h$,

$$\frac{d}{dx}[h(x)]^n = n[h(x)]^{n-1}\frac{d}{dx}[h(x)]$$

Example

Differentiate the following function and simplify the answer.

$$f(x) = (t^4 - 4t^2 + 4)^6$$
Example
Differentiate the following function and simplify the answer.

\[ f(x) = (2x + 1)^4(3x - 5)^2 \]
Combination with other rules

Example
Differentiate the following function and simplify the answer.

\[ F(x) = \frac{(1 - 2x)^3}{(3x + 1)^2} \]
Higher derivatives

Example
Find the second derivative of the given function

\[ y = (1 - x^2)^3 \]