1.6. One-sided Limits and Continuity

One-sided Limit
If \( f(x) \) approaches \( L \) as \( x \) tends toward \( c \) from the left \((x < c)\), we write
\[
\lim_{{x \to c^-}} f(x) = L.
\]
Likewise, if \( f(x) \) approaches \( M \) as \( x \) tends toward \( c \) from the right \((x > c)\), then
\[
\lim_{{x \to c^+}} f(x) = M.
\]

Example
Find \( \lim_{{x \to 2^-}} f(x) \) and \( \lim_{{x \to 2^+}} f(x) \) for the function
\[
f(x) = \frac{x^2 + 3}{x - 2}
\]
One-sided Limit

Example

Find \( \lim_{x \to -1^-} f(x) \) and \( \lim_{x \to -1^+} f(x) \) for the function

\[
f(x) = \begin{cases} 
  2 & \text{if } x < -1 \\
  \frac{2}{x - 1} & \text{if } x \geq -1 \\
  x^2 - x & \text{if } x \geq -1 
\end{cases}
\]
Existence of a Limit

Theorem

The two-sided limit \( \lim_{x \to c} f(x) \) exists if and only if the two one-sided limits \( \lim_{x \to c^-} f(x) \) and \( \lim_{x \to c^+} f(x) \) both exist and are equal, and then

\[
\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)
\]

Example

Determine whether \( \lim_{x \to 1} f(x) \) exists, where

\[
f(x) = \begin{cases} 
2x + 1 & \text{if } x < 1 \\
-x^2 + 2x + 2 & \text{if } x \geq 1
\end{cases}
\]
Continuity

A function $f$ is **continuous** at $c$ if all three of these conditions are satisfied:

a. $f(c)$ is defined
b. $\lim_{x \to c} f(x)$ exists
c. $\lim_{x \to c} f(x) = f(c)$

If $f(x)$ is not continuous at $c$, it is said to have a **discontinuity** there.

**Example**

Decide if $f(x) = x^3 - x^2 + x - 4$ is continuous at $x = 0$. 
Continuity

Example

Decide if \( f(x) = \frac{2x + 5}{2x - 4} \) is continuous at \( x = 2 \).
Continuity of Polynomials and Rational Functions

A polynomial or a rational function is continuous wherever it is defined.

Example

List all values of $x$ for which $f(x)$ is not continuous

$$f(x) = \frac{x^2 - 2x + 1}{x^2 - x - 2}$$
Example

Decide if \( f(x) = \begin{cases} 
  x + 1 & \text{if } x < 0 \\
  x - 1 & \text{if } x \geq 0 
\end{cases} \) is continuous at \( x = 0 \).
Example
Find the value of the constant $A$ such that the function

$$f(x) = \begin{cases} 
1 - 2x & \text{if } x < 2 \\
Ax^2 + 2x - 3 & \text{if } x \geq 2
\end{cases}$$

will be continuous for all $x$. 
Continuity on an Interval

A function \( f(x) \) is said to be continuous on an open interval \( a < x < b \) if it is continuous at each point \( x = c \) in that interval. Moreover, \( f \) is continuous on the closed interval \( a \leq x \leq b \) if it is continuous on the open interval \( a < x < b \) and

\[
\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b)
\]

Example
Discuss the continuity of

\[
f(x) = \begin{cases} 
  x^2 - 3x & \text{if } x < 2 \\
  4 + 2x & \text{if } x \geq 2
\end{cases}
\]
on the open interval \( 0 < x < 2 \) and the closed interval \( 0 \leq x \leq 2 \).
The intermediate value property

If $f(x)$ is continuous on the interval $a \leq x \leq b$ and $L$ is a number between $f(a)$ and $f(b)$, the $f(c) = L$ for some number $c$ between $a$ and $b$. In other words, a continuous function attains all values between any two of its values.

Example

Show that the equation $\sqrt[3]{x} = x^2 + 2x - 1$ must have at least one solution on the interval $0 \leq x \leq 1$. 