Math 108, Solution of Midterm Exam 3

1. Find an equation of the tangent line to the curve \( x^3 + y^3 = 2xy \) at the point \((1, 1)\).

Solution. Differentiating both sides of the given equation with respect to \(x\), we get

\[
\frac{d}{dx} [x^3 + y^3] = \frac{d}{dx} [2xy]
\]

\[
3x^2 + 3y^2 \frac{dy}{dx} = 2y \frac{d}{dx} (x) + 2x \frac{d}{dx} (y)
\]

\[
3x^2 + 3y^2 \frac{dy}{dx} = 2y + 2x \frac{dy}{dx}
\]

\[
(3y^2 - 2x) \frac{dy}{dx} = 2y - 3x^2
\]

\[
\frac{dy}{dx} = \frac{2y - 3x^2}{3y^2 - 2x}
\]

So, the slope of the tangent line is

\[
\left. \frac{dy}{dx} \right|_{(1, 1)} = \frac{2(1) - 3(1)^2}{3(1)^2 - 2(1)} = -1
\]

and hence the equation of the tangent line is

\[
y - 1 = -1(x - 1)
\]

\[
y = -x + 2
\]

2. Let \( f \) be a function defined by \( f(x) = x^5 - 5x^4 + 3 \).

(a) Find intervals on which \( f \) is increasing and decreasing.

Solution. The derivative of \( f \) is

\[
f'(x) = \frac{d}{dx} [x^5 - 5x^4 + 3]
\]

\[
= 5x^4 - 20x^3
\]

\[
= 5x^3(x - 4)
\]

which is continuous everywhere, with \( f'(x) = 0 \) where \( x = 0 \) and \( x = 4 \). The numbers 0 and 4 divide the \( x \) axis into three open intervals; namely, \( x < 0 \), \( 0 < x < 4 \), and \( x > 4 \). Choose a test number from each of these intervals; say \( c = -1 \) from \( x < 0 \), \( c = 1 \) from \( 0 < x < 4 \), and \( c = 5 \) from \( x > 4 \). Then, evaluate \( f'(c) \) for each test number:

\[
f'(-1) = 5(-1)^3(-1 - 4) > 0,
\]

\[
f'(1) = 5(1)^3(1 - 4) < 0,
\]

\[
f'(5) = 5(5)^3(5 - 4) > 0.
\]

So, \( f(x) \) is increasing on \( x < 0 \) and \( x > 4 \) and decreasing on \( 0 < x < 4 \).
(b) Find \( x \)-coordinates of all critical points of \( f \) and classify each of them as a relative maximum, a relative minimum, or neither.

**Solution.** Since \( f'(x) \) exists for all \( x \), the only critical numbers are where \( f'(x) = 0 \), i.e., \( x = 0 \) and \( x = 4 \). Since \( f'(x) > 0 \) to the left of \( x = 0 \) and \( f'(x) < 0 \) to the right of \( x = 0 \), the critical point where \( x = 0 \) is a relative maximum. Since \( f'(x) < 0 \) to the left of \( x = 4 \) and \( f'(x) > 0 \) to the right of \( x = 4 \), the critical point where \( x = 4 \) is a relative minimum.

(c) Find intervals on which the graph of \( f \) is concave up and concave down.

**Solution.** Since the first derivative of \( f \) is
\[
f'(x) = 5x^4 - 20x^3,
\]
the second derivative of \( f \) is
\[
f''(x) = 20x^3 - 60x^2
\]
\[= 20x^2(x - 3).
\]
The second derivative \( f''(x) \) is continuous for all \( x \) and \( f''(x) = 0 \) for \( x = 0 \) and \( x = 3 \). These numbers divide the \( x \) axis into three intervals; namely \( x < 0 \), \( 0 < x < 3 \), and \( x > 3 \). Evaluating \( f''(x) \) at test numbers in each of these intervals (say, \( x = -1 \), \( x = 1 \), and \( x = 4 \), respectively), we find
\[
f''(-1) = 20(-1)^2(-1 - 3) < 0,
\]
\[
f''(1) = 20(1)^2(1 - 3) < 0,
\]
\[
f''(4) = 20(4)^2(4 - 3) > 0.
\]
Thus, the graph of \( f(x) \) is concave down for \( x < 0 \) and \( 0 < x < 3 \) and concave up for \( x > 3 \).

(d) Find \( x \)-coordinates of all inflection points of \( f \).

**Solution.** Since the concavity does not change at \( x = 0 \), \( f(x) \) does not an inflection point at \( x = 0 \). Since the concavity changes from downward to upward at \( x = 3 \), the graph of \( f(x) \) has an inflection point at \( x = 3 \).

(3) Let \( f \) be a function defined everywhere except \( x = 1 \) and whose derivative is given by
\[
f'(x) = \frac{x^2 - 7x + 10}{x - 1}.
\]

(a) Find all critical numbers of \( f \) and classify each of them as a relative maximum, a relative minimum, or neither.
Solution. Since
\[
f'(x) = \frac{x^2 - 7x + 10}{x - 1} = \frac{(x - 2)(x - 5)}{x - 1}
\]
is continuous everywhere except \(x = 1\) which is not in the domain of \(f(x)\) and \(f'(x) = 0\) when \(x = 2\) or \(x = 5\), the critical numbers are \(x = 2\) and \(x = 5\). The critical numbers 2 and 5 together with \(x = 1\) divide the \(x\) axis into four open intervals; namely, \(x < 1\), \(1 < x < 2\), \(2 < x < 5\), and \(x > 5\). Evaluating \(f'(x)\) at test numbers in each of these intervals (say, \(x = 0\), \(x = \frac{3}{2}\), \(x = 3\), and \(x = 6\), respectively), we find
\[
\begin{align*}
f'(0) &= \frac{(0 - 2)(0 - 5)}{0 - 1} < 0, \\
f'(\frac{3}{2}) &= \frac{\left(\frac{3}{2} - 2\right)(\frac{3}{2} - 5)}{\frac{3}{2} - 1} > 0, \\
f'(3) &= \frac{(3 - 2)(3 - 5)}{3 - 1} < 0, \\
f'(6) &= \frac{(6 - 2)(6 - 5)}{6 - 1} > 0.
\end{align*}
\]
Since \(f'(x) > 0\) to the left of \(x = 2\) and \(f'(x) < 0\) to the right of \(x = 2\), the critical point where \(x = 2\) is a relative maximum. Since \(f'(x) < 0\) to the left of \(x = 5\) and \(f'(x) > 0\) to the right of \(x = 5\), the critical point where \(x = 5\) is a relative minimum.

(b) Find \(x\)-coordinates of all inflection points of \(f\).

Solution. By the quotient rule,
\[
f''(x) = \frac{(x - 1)\frac{d}{dx}(x^2 - 7x + 10) - (x^2 - 7x + 10)\frac{d}{dx}(x - 1)}{(x - 1)^2}
\]
\[
= \frac{(x - 1)(2x - 7) - (x^2 - 7x + 10)(1)}{(x - 1)^2}
\]
\[
= \frac{2x^2 - 7x - 2x + 7 - x^2 + 7x - 10}{(x - 1)^2}
\]
\[
= \frac{x^2 - 2x - 3}{(x - 1)^2}
\]
\[
= \frac{(x + 1)(x - 3)}{(x - 1)^2}
\]
Since \(f''(x)\) is not continuous at \(x = 1\) and \(f''(x) = 0\) when \(x = -1\) or \(x = 3\), the \(x\)-axis is divided into four intervals; namely, \(x < -1\), \(-1 < x < 1\), \(1 < x < 3\), and \(x > 3\). Evaluating \(f''(x)\) at test
numbers in each of these intervals (say, $x = -2$, $x = 0$, $x = 2$, and $x = 4$, respectively), we find

\[
f'(-2) = \frac{(-2 + 1)(-2 - 3)}{(-2 - 1)^2} > 0, \\
f'(0) = \frac{(0 + 1)(0 - 3)}{(0 - 1)^2} < 0, \\
f'(2) = \frac{(2 + 1)(2 - 3)}{(2 - 1)^2} < 0, \\
f'(4) = \frac{(4 + 1)(4 - 3)}{(4 - 1)^2} > 0.
\]

Since the concavity changes from upward to downward at $x = -1$, the graph of $f(x)$ has an inflection point at $x = -1$. Since the concavity changes from downward to upward at $x = 3$, the graph of $f(x)$ has an inflection point at $x = 3$.

**4** Find all vertical and horizontal asymptotes of the graph of the function \( \frac{2x^2 + x - 3}{x^2 + x - 2} \).

**Solution.** Since the denominator of the given function can be factored into \((x - 1)(x + 2)\), it is

0 when $x = 1$ or $x = -2$. Since

\[
\lim_{x \to 1^-} \frac{2x^2 + x - 3}{x^2 + x - 2} = \lim_{x \to 1^-} \frac{(x - 1)(2x + 3)}{(x - 1)(x + 2)} = \lim_{x \to 1^-} \frac{2x + 3}{x + 2} = \frac{5}{3},
\]

and

\[
\lim_{x \to 1^+} \frac{2x^2 + x - 3}{x^2 + x - 2} = \lim_{x \to 1^+} \frac{(x - 1)(2x + 3)}{(x - 1)(x + 2)} = \lim_{x \to 1^+} \frac{2x + 3}{x + 2} = \frac{5}{3},
\]

$x = 1$ is not a vertical asymptote. Since

\[
\lim_{x \to -2^+} \frac{2x^2 + x - 3}{x^2 + x - 2} = \lim_{x \to -2^+} \frac{(x - 1)(2x + 3)}{(x - 1)(x + 2)} = \lim_{x \to -2^+} \frac{2x + 3}{x + 2} = -\infty
\]

$x = -2$ is a vertical asymptote of the given graph. Therefore $x = -2$ is the only vertical asymptote of the given function.

Since

\[
\lim_{x \to +\infty} \frac{2x^2 + x - 3}{x^2 + x - 2} = \lim_{x \to +\infty} \frac{2 + \frac{1}{x} - \frac{3}{x^2}}{1 + \frac{1}{x} - \frac{1}{x^2}} = 2
\]

and

\[
\lim_{x \to -\infty} \frac{x^2 + 2x - 3}{x^2 + x - 2} = \lim_{x \to -\infty} \frac{2 + \frac{1}{x} - \frac{3}{x^2}}{1 + \frac{1}{x} - \frac{1}{x^2}} = 2
\]

$y = 2$ is a horizontal asymptote of the given graph.
Find the absolute maximum and absolute minimum (if any) of the function \( f(x) = x^3 - 3x^2 - 9x + 10 \) on the interval \(-2 \leq x \leq 2\).

**Solution.** From the derivative
\[
f'(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)
\]
we see that the critical numbers are \( x = -1 \) and \( x = 3 \). Of these, only \( x = -1 \) lies in the interval \(-2 < x < 2\). Compute \( f(x) \) at the critical number \( x = -1 \) and at endpoints \( x = -2 \) and \( x = 2 \).

\[
egin{align*}
f(-1) &= (-1)^3 - 3(-1)^2 - 9(-1) + 10 = 15 \\
f(-2) &= (-2)^3 - 3(-2)^2 - 9(-2) + 10 = 8 \\
f(2) &= (2)^3 - 3(2)^2 - 9(2) + 10 = -12
\end{align*}
\]

Compare these values to conclude that the absolute maximum of \( f(x) \) on the interval \(-1 \leq x \leq 3\) is \( f(-1) = 15 \) and the absolute minimum is \( f(2) = -12 \).

Find the absolute maximum and absolute minimum (if any) of the function \( f(x) = x^3 - 12x + 20 \) on the interval \( x \geq 0 \).

**Solution.** Since \( f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2) \), \( x = 2 \) is the only critical number of \( f(x) \) on \( x \geq 0 \). Since \( f'(x) < 0 \) for \( 0 < x < 2 \) and \( f'(x) > 0 \) for \( x > 2 \), the graph of \( f \) is decreasing for \( 0 < x < 2 \) and increasing for \( x > 2 \). It follows that \( f(2) = (2)^3 - 12(2) + 20 = 4 \) is the absolute minimum of \( f \) on the interval \( x \geq 0 \) and that there is no absolute maximum.