1. (10 pts) The graph below depicts some solutions to the ODE $x' = t - e^{-x}$. It appears that there are multiple solutions through the points A, B, and C. Is this possible? Explain.

Multiple solutions through A, B or C are impossible! This is because $f(t,x) = t - e^{-x}$ is continuously differentiable w.r.t $x$ everywhere in the $tx$-plane. In particular,

$$f_x(t,x) = -e^{-x}(-1) = e^{-x}$$

is continuous at all pts $(t,x) \in \mathbb{R}^2$. It follows from the FTU that solutions through A, B, and C are unique; that is, there can't be two or more solutions through any point.

(The initial resolutions of the printing process cannot allow for the physical separation of the solutions when they are very close together.)
2. (17 pts) Consider the autonomous ODE \( x' = x^2 (x^2 - 4) (x^2 + 1) \).

(a) (3 pts) Determine all of the equilibrium solutions.
\[
x^2 (x^2 - 4)/(x + 1) = 0 \Rightarrow x = 0, \pm 2, \pm i \]
Equilibria: \( x = 0, -2, 2 \)

(b) (2 pts) Which of the equilibria are attractors?
From graph of \( f(x) \), get \( x = -2 \)

(c) (2 pts) Which of the equilibria are repellers?
From graph of \( f(x) \), get \( x = 2 \)

(d) (5 pts) Graph the phase line.

(e) (2 pts) Sketch all of the equilibrium solutions in the \( tx \)-plane below.

(f) (3 pts) These equilibrium solutions divide the \( tx \)-plane into regions. Sketch at least one solution in each of these regions.
3. (18 pts) Which one of the following best represents the direction field of ODE $x'' - x' - x = 0$?

Express and system:
Set $y = x'$
Then $y'' = x'' = x' + x$

Set system:
$\begin{cases} x' = y \\ y' = x + y \end{cases}$

at $(1, 0)$
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

implies

at $(0, 1)$
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Just $(c)$ is left, check ad other point $b$.

at $(1, 1)$
$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(c)
4. (50 pts) Consider the ODE $x'' + 3x' + 2x = e^{-t}$.

(a) (20 pts) Use the UC method to calculate a particular solution.

Char poly: $\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = -1$.

$FSS = \{e^{-2t}, e^{-t}\}$.

$P_f = \{e^{-t}\}$, since $P_f$ is a subset of $FSS$.

Then modify $P_f$ by $P_f' = \{te^{-t}\}$. The trial soln is

$x_p = Ae^{-t}$

$x_p' = Ae^{-t} - Ate^{-t}$

$x_p'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}$

Substitute into ODE:

$-2Ae^{-t} + Ate^{-t} + 3(Ae^{-t} - Ate^{-t}) + 2Ae^{-t} = e^{-t}$

$-2Ae^{-t} + Ate^{-t} + 3Ae^{-t} - 3Ate^{-t} + 2Ae^{-t} = e^{-t}$

$\Rightarrow \ Ae^{-t} = e^{-t}$

$\Rightarrow \ A = 1$.

So $x_p = te^{-t}$

(b) (20 pts) Use the VP method to calculate a particular solution.

Set $\phi_1(t) = e^{-2t}$

$\phi_2(t) = e^{-t}$

$W(t) = \det \begin{bmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{bmatrix} = -e^{-3t} + 2e^{-3t} = e^{-3t}$

$c_1(t) = \int \frac{-e^{-2t} e^{-t}}{e^{-3t}} dt = \int e^t dt = e^t$

$c_2(t) = \int \frac{e^{-2t} e^{-t}}{e^{-3t}} dt = \int e^t dt = t$

$x_p = c_1(t) \phi_1(t) + c_2(t) \phi_2(t) = (e^t) e^{-2t} + t e^{-t} = -e^{-t} + te^{-t}$

$= (t-1) e^{-t}$

(c) (10 pts) Solve the IVP $x'' + 3x' + 2x = e^{-t}$, $x(0) = 1$, $x'(0) = 0$.

$x_p = c_1 e^{-2t} + c_2 e^{-t} + (t-1) e^{-t}$

$\Rightarrow \ i = c_1 + c_2 - 1$

$x_p' = -2c_1 e^{-2t} - c_2 e^{-t} + e^{-t} - (t-1) e^{-t}$

$\Rightarrow \ 0 = -2c_1 - c_2 + 1 + 1$

$c_1 + c_2 = 2$

$2c_1 + c_2 = 2$

$\Rightarrow \ (Subtract \ i^{th} \ eqn \ from \ the \ 2^{nd}) \ \ c_1 = 0, \ c_2 = 2$.

Get:

$x = 2e^{-t} + (t-1) e^{-t} = 2e^{-t} + te^{-t} - e^{-t}$

$= e^{-t} + te^{-t} = (t+1)e^{-t}$
5. (30 pts) Consider the IVP $x'' + x = f(t), x(0) = 0, x'(0) = 0$, where
\[
f(t) = \begin{cases} 
0 & \text{if } -\infty < t < 0 \\
1 & \text{if } 0 \leq t < 2\pi \\
0 & \text{if } 2\pi \leq t < \infty
\end{cases}
\]

See Example 16.5

Done in class

(a) (25 pts) Use the Green's kernel solution to solve the IVP.

First poly: $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$. Then $K(t, \tau) = \sin(t - \tau)$

Take $\tau = 0$. Then $x_p = \int_0^\tau \sin(t - \tau) f(\tau) d\tau = \int_0^\tau \sin(t - \tau) f(\tau) d\tau$

There are 3 cases:

(i) $t < 0$. Then $f(t) = 0$ whenever $t < 0$, so

\[x_p = \int_0^t \sin(t - \tau) \cdot 0 \, d\tau = 0\]

(ii) $0 \leq t < 2\pi$. Then $f(t) = 1$ whenever $0 \leq t < 2\pi$, so

\[x_p = \int_0^t \sin(t - \tau) \cdot 1 \, d\tau = \int_0^t \sin(t - \tau) \, d\tau\]

In order to change variables, set $\tau = t - \nu$. Then $d\tau = -d\nu$. When $\tau = 0$, $\nu = t$, and when $\tau = t$, $\nu = 0$

\[x_p = \int_0^t \sin(t - \nu) (-d\nu) = \int_0^t \sin \nu \, d\nu = -\cos \nu \Big|_0^t = -(\cos t - \cos 0) = 1 - \cos t\]

(iii) $2\pi \leq t < \infty$. Then $f(t) = 0$ for all $t \geq 2\pi$. Thus

\[x_p = \int_0^{2\pi} \sin(t - \nu) \cdot 0 \, d\nu + \int_0^t \sin(t - \nu) \cdot 0 \, d\nu\]

Since $d\nu = -d\nu$, change variables in the integral: $u = t - \nu$, so $d\nu = -du$, and

\[x_p = \int_0^t \sin u \cdot 1 \, (-du) = \int_{-t}^0 \sin u \, du = -\cos u \Big|_{-t}^0 = -\cos t + \cos (t - 2\pi) = 0 \quad \text{[since} \cos(t - 2\pi) = \cos t]\]

Thus,

\[x_p = \begin{cases} 
0, & -\infty < t < 0 \\
1 - \cos t, & 0 \leq t < 2\pi \\
0, & 2\pi \leq t < \infty
\end{cases}\]

(b) (5 pts) Sketch the solution on the interval $-2\pi \leq t \leq 4\pi$.

Since $x(0) = x = 0$, $x'(0) = x = 0$, then the solution to the IVP is $x = x_p$

\[x = \begin{cases} 
0, & -\infty < t < 0 \\
1 - \cos t, & 0 \leq t < 2\pi \\
0, & 2\pi \leq t < \infty
\end{cases}\]