MATH 214-001 - Exam 1, 23 Feb, 2009

- **PARTIAL CREDIT**: To receive any partial credit you must show all of your work.
- **CALCULATORS, COMPUTERS**: not allowed
- **TABLE OF INTEGRALS** will be provided - please return after the exam
- **CHEAT SHEET**: Both sides of a 3×5 "cheat sheet" is permitted - no other materials allowed
- **GMU HONOR CODE**: Your work on this exam is governed by the code
- There are 115 possible points on this exam. Think of the 15 points in excess of 100 as **bonus** points.

1. (32 pts) Consider the ODE \( x^2 x' = \frac{1}{3} \)

   (a) (6 pts) Sketch an appropriate direction field in the \( tx \)-window \(-2 \leq t \leq 2, -2 \leq x \leq 2\).

   \[ x' = \frac{1}{3x^2} \]

   (b) (8 pts) Use the direction field to sketch the solution with initial value \( x(0) = 1 \).

   (c) (12 pts) Now compute an explicit solution \( x = \phi(t) \) with the initial value \( x(0) = 1 \).

   \[
   \int x^2 \, dx = \frac{1}{3} \int dt
   \]

   \[
   \Rightarrow \frac{1}{3} x^3 = \frac{1}{3} t + C \quad \Rightarrow \quad x^3 = t + 3C
   \]

   \[
   x(0) = 1 \Rightarrow 1 = 3C \Rightarrow C = \frac{1}{3} \quad \Rightarrow \quad x^3 = t + 1
   \]

   \[
   x = \sqrt[3]{t + 1}
   \]

   (d) (6 pts) Determine the maximal (i.e., largest) interval of definition of \( \phi \)? Explain your answer.

   \[
   \text{cannot allow } t = -1, \quad \text{otherwise } x = 0 \text{ and } x' \text{ is undefined. Then } \]

   \[
   I = (-1, \infty)
   \]
2. (25 pts) Consider the ODE \[ \frac{dx}{dt} = \frac{e^t + x}{2 \sin x - t} \]

(a) (15 pts) Determine a general implicit solution of the form \( \Phi(t, x) \).

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Rewrite \((2 \sin x - t)x' = e^{t}+x\) or \(e^{t}+x + (t-2 \sin x)x' = 0\)
Set \(M = e^{t}+x\) \(\Rightarrow M_x = 1\)
\(N = t-2 \sin x\) \(\Rightarrow N_t = 1\)

\(\Phi(t,x) = \int M \, dt = \int (e^{t}+x) \, dt = e^{t}+tx + C(t, x)\)
From \(M_x = N\) we get \(\frac{\partial}{\partial x} [e^{t}+tx + C(t, x)] = t-2 \sin x\)

\(\Phi(t,x) = e^{t}+tx + 2 \cos x + C_0\)
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Some of the level curves of the surface defined by the general (implicit) solution to part (a) are displayed below.

(b) (5 pts) Darken precisely the graph of the solution to the ODE that satisfies the initial data \(x(1.3) = -5\).

(c) (5 pts) Estimate the maximum interval of definition by "eyeballing" the graph you identified in part (a).

\[0.8 < t < 1.8\]
3. (18 pts) Match the slope fields below to the following ODEs:
(a) \( x' = t + x \)  
(b) \( x' = \frac{1}{4}t^2 \)  
(c) \( x' = \cos x \)  
(d) \( x' = 2x/t \)  
(e) \( x' = xe^{-t} \)  
(f) \( x' = 1 - tx \)
Mark each answer on the appropriate slope field.
4. (40 pts) Consider the ODE \( x' + x = g(t) \), where

\[
g(t) = \begin{cases} 
1, & \text{if } -\infty < t < 1 \\
0, & \text{if } 1 \leq t < \infty
\end{cases}
\]

(a) (10 pts) Sketch a slope field of the ODE on the grid below.

(b) (5 pts) Use your slope field to sketch the solution curve on the interval \( 0 \leq t \leq 4 \) with initial condition \( x(0) = 1 \).

(c) (25 pts) Calculate an explicit solution to the ODE using the initial condition \( x(0) = 1 \). In particular, determine a (piecewise) formula for the solution on the interval \( 0 \leq t \leq 4 \). Use either the integral formula or piecewise method.

\[
\begin{align*}
\text{Integral Formula Solution:} & \quad e^{P(t)} = e^t, \\
x(t) & = e^{-P(t)} \left[ x_0 + \int_{t_0}^{t} e^{P(\tau)} g(\tau) d\tau \right].
\end{align*}
\]

\[
g(t) = \begin{cases} 
1, & t < 1 \\
0, & t \geq 1
\end{cases}
\]

\[
x(t) = \begin{cases} 
1, & t < 1 \\
1 + \int_0^t e^{-\tau} d\tau, & t \geq 1
\end{cases}
\]

\[
x(t) = \begin{cases} 
1, & t < 1 \\
1 + e^{t-1}, & t \geq 1
\end{cases}
\]
Piecewise Methods

(i) \(-\infty < t < 1\):

IVP: \(x' + x = 1, \quad x(0) = 1\)

Solve by methods of Sec. 4: \(u(t) = e^t\), \(e^t x' + e^t x = e^t\cdot 1\)

\[
\frac{d}{dt} [e^t x] = e^t \quad \Rightarrow \quad e^t x = \int e^t dt = e^t + c
\]

\[
\Rightarrow \quad x = 1 + ce^{-t}
\]

\(x(0) = 1 \Rightarrow 1 = 1 + ce^0 \Rightarrow c = 0\)

\(\Rightarrow x(t) = 1, \quad \infty < t < 1\).

(ii) \(1 \leq t < \infty\). The ODE has changed since \(g(t) = 0\) when \(t \geq 1\). Thus ODE is \(x' + x = 0\).

Also, we need the initial data at \(t = 1\) (not \(t = 0\)).

For \(x(1)\) take: \(\lim_{t \to 1^-} x(t) = 1\) from (i).

The solution to the first segment at its right endpoint. So new IVP is \(x' + x = 0, \quad x(1) = 1\).

Again \(u(t) = e^t\), so

\[
e^t x' + e^t x = e^t \cdot 0
\]

\[
\frac{d}{dt} [e^t x] = 0 \quad \Rightarrow \quad e^t x = \int 0 dt = c
\]

\[
x = ce^{-t}
\]

\(x(1) = 1 \Rightarrow 1 = ce^{-1} \Rightarrow c = e^2\). So,

\[
x = e^2 e^{-t} = e^{(2-t)} \quad \text{when} \quad t \geq 1.
\]

Piece together:

\[
x = \begin{cases} 
1, & \infty < t < 1 \\
\frac{1}{e^{(t-1)}}, & 1 \leq t < \infty
\end{cases}
\]