Computing Dimension of Strange Attractors

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Outline

- Define Strange Attractor
- Examples
- Define Box-Counting Dimension
- Why study dimension of attractors?
- How to compute dimension of attractors
- Computing the Dimension of Hénon Attractor
- Open Questions

Definition of an Attractor

- An attractor is a subset A of the phase space such that:
  - A is invariant under f, do if x is an element of A then so is f(x).
  - There is a neighborhood of A, B(A) called the basin of attraction for A, such that B(A) = {x| for all neighborhoods S of A there is a N so that for all n>N $f^n(x)$ in S}. In other words, B(A) is the set of points that 'enter A in the limit'.
  - There is no proper subset of A with the first two properties.

Types of Attractors

- Fixed point Attractor
- Periodic Attractor
- Strange Attractor – an attractor with non-integer dimension. Often, strange attractors have a local topological structure that is a product of a submanifold and a Cantor-like set.

Examples of Strange Attractors

- Lorenz Attractor
- Rössler Attractor
- Hénon Attractor

Lorenz Attractor

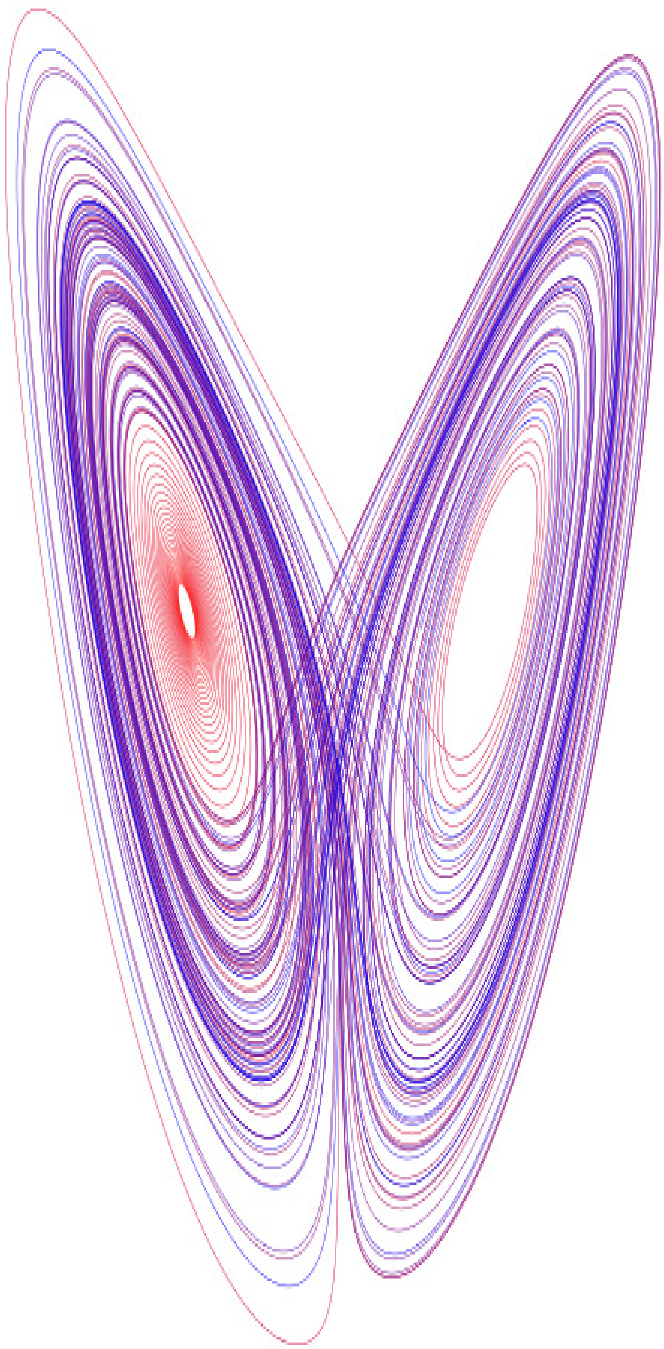
- Introduced by Edward Lorenz in 1963.
- Well known for butterfly structure.
- Originally derived from equations of convection in the atmosphere, but same dynamics can arise in lasers.

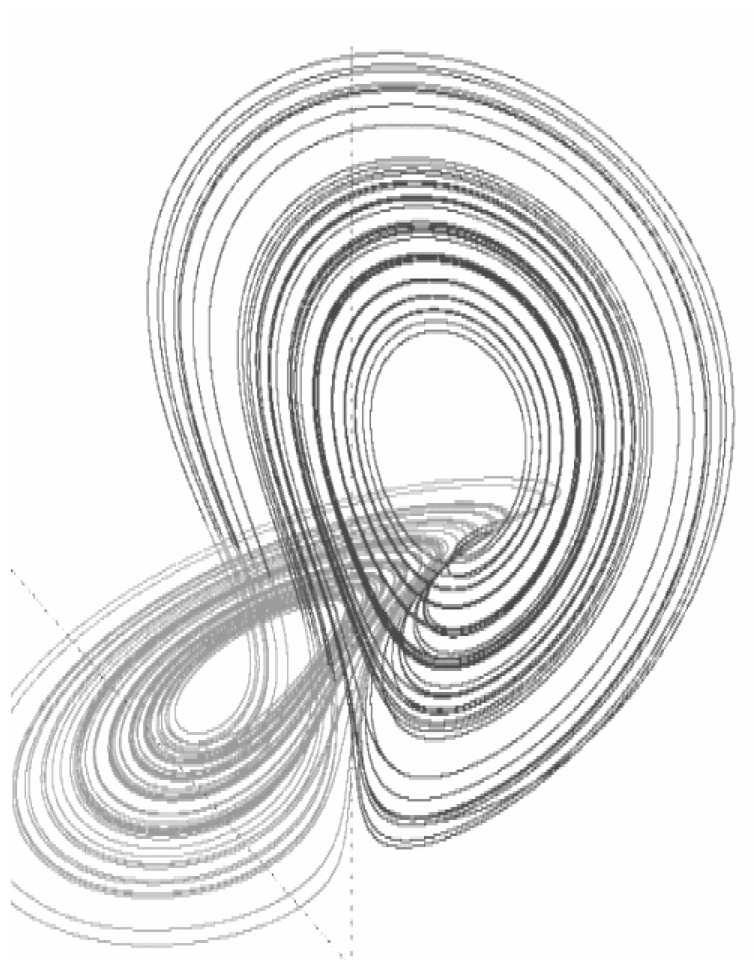
$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

Lorenz Attractor





www.wikipedia.com

www.pha.jhu.edu/~ldb/seminar/attractors.html

Rössler Attractor

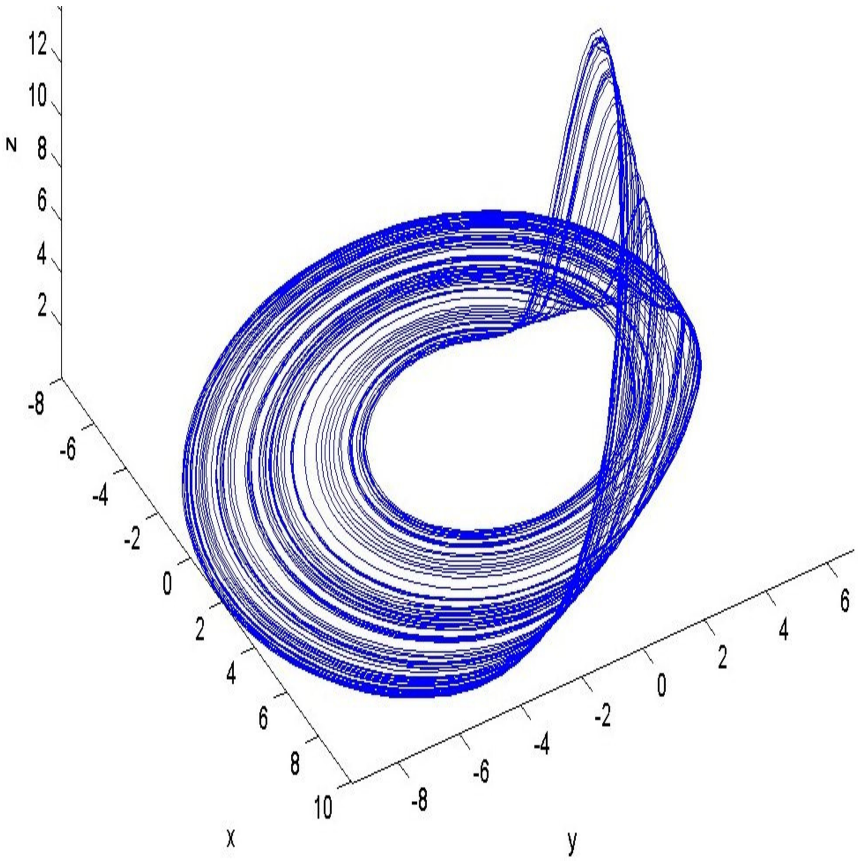
- The Rössler system initially was intended to be a system whose dynamics were similar to the Lorenz system, but were more easy to analyze.

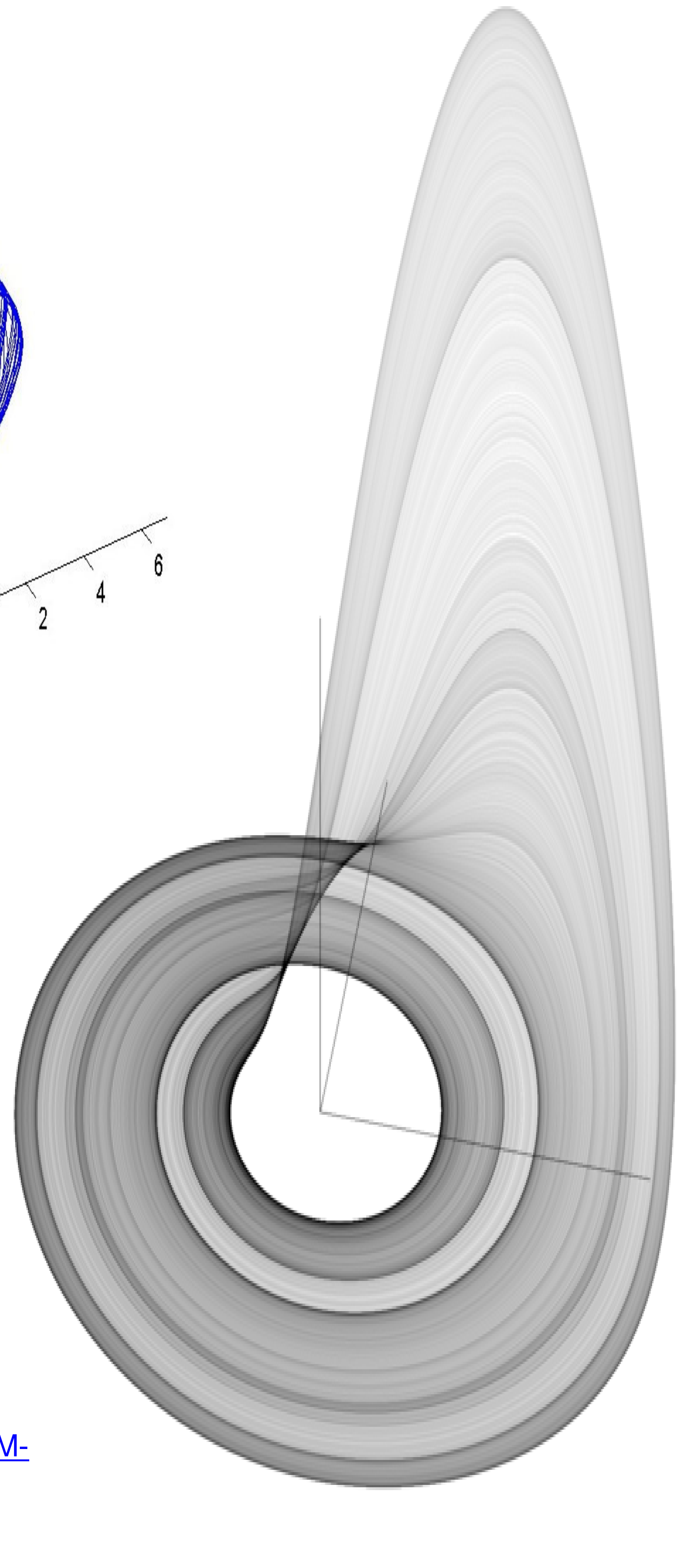
$$\frac{dx}{dt} = -y - z$$

$$\frac{dy}{dt} = x + ay$$

$$\frac{dz}{dt} = b + z(x - c)$$

Rössler Attractor





www.wikipedia.com

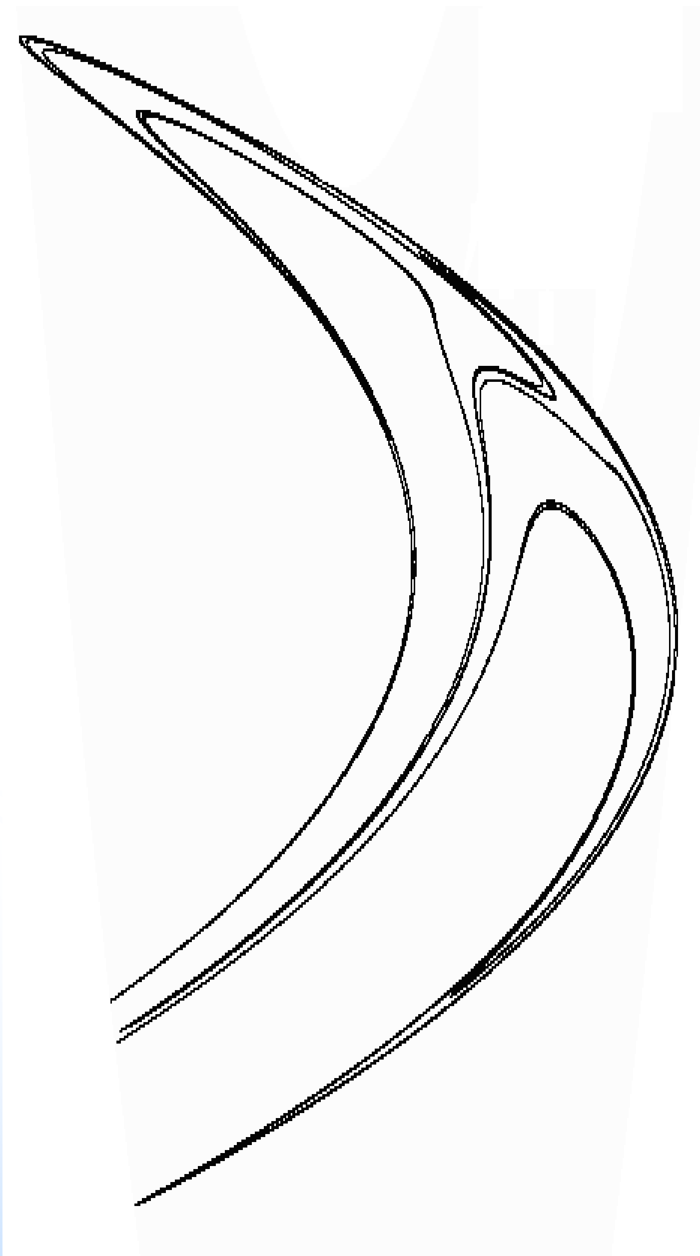
http://www-rohan.sdsu.edu/~rcarrete/teaching/M-638/lectures/lectures.html

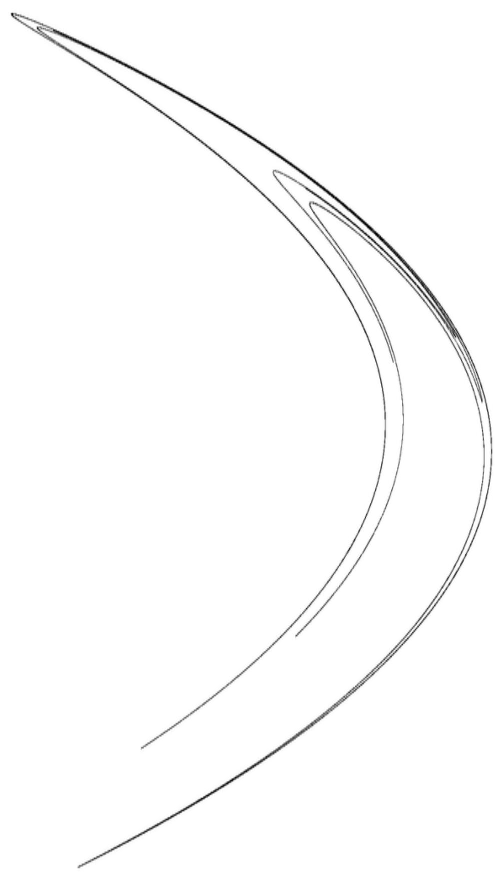
Hénon Attractor

$$H: (x,y) \mapsto (1 - ax^2 + by, x)$$

- Introduced by Michel Hénon, a French astronomer, in 1975.
- The Hénon map is a discrete-time 2-dimensional map, which makes analyzing it and simulating it on a computer much easier.

Hénon Attractor





sprott.physics.wisc.edu/chaos/henondky.htm

Fractal Dimension

- A fractal is a self similar set that is invariant under scaling and is "too irregular to be easily described in traditional Euclidean Geometric language".
- Fractal dimension can give us an idea of how the fractal fills space and can also give us a "rate of scaling" of the fractal.

Box-Counting Dimension

- $$Boxdim(S) = \lim_{\varepsilon \to 0} \frac{\ln(N(1/\varepsilon))}{\ln(1/\varepsilon)}$$

for a set S

where ε is the size of the "box"

and N(1/ε) is the number of boxes required to cover the set S.

Example

- Unit Interval:

  If $\varepsilon = 1/k$, then we need k boxes to cover the interval, so

  $$\text{boxdim}([0,1]) = \lim_{k\to\infty} \frac{\ln(k)}{\ln(k)} = 1$$

- Unit Square:

  If $\varepsilon = 1/k$, then we need k^2 $\varepsilon \times \varepsilon$ boxes to cover the square, so

  $$\text{boxdim}([0,1]\times[0,1]) = \lim_{k\to\infty} \frac{\ln(k^2)}{\ln(k)} = 2$$

Why Study Dimension of Attractors?

- Those who study dynamical systems believe there is a relationship between the properties and behavior of a dynamical system and the topology of the attractor associated with that dynamical system.
- We study dimension of attractors as a tool for understanding the topology of the attractor.

How we compute box-counting dimension of attractors

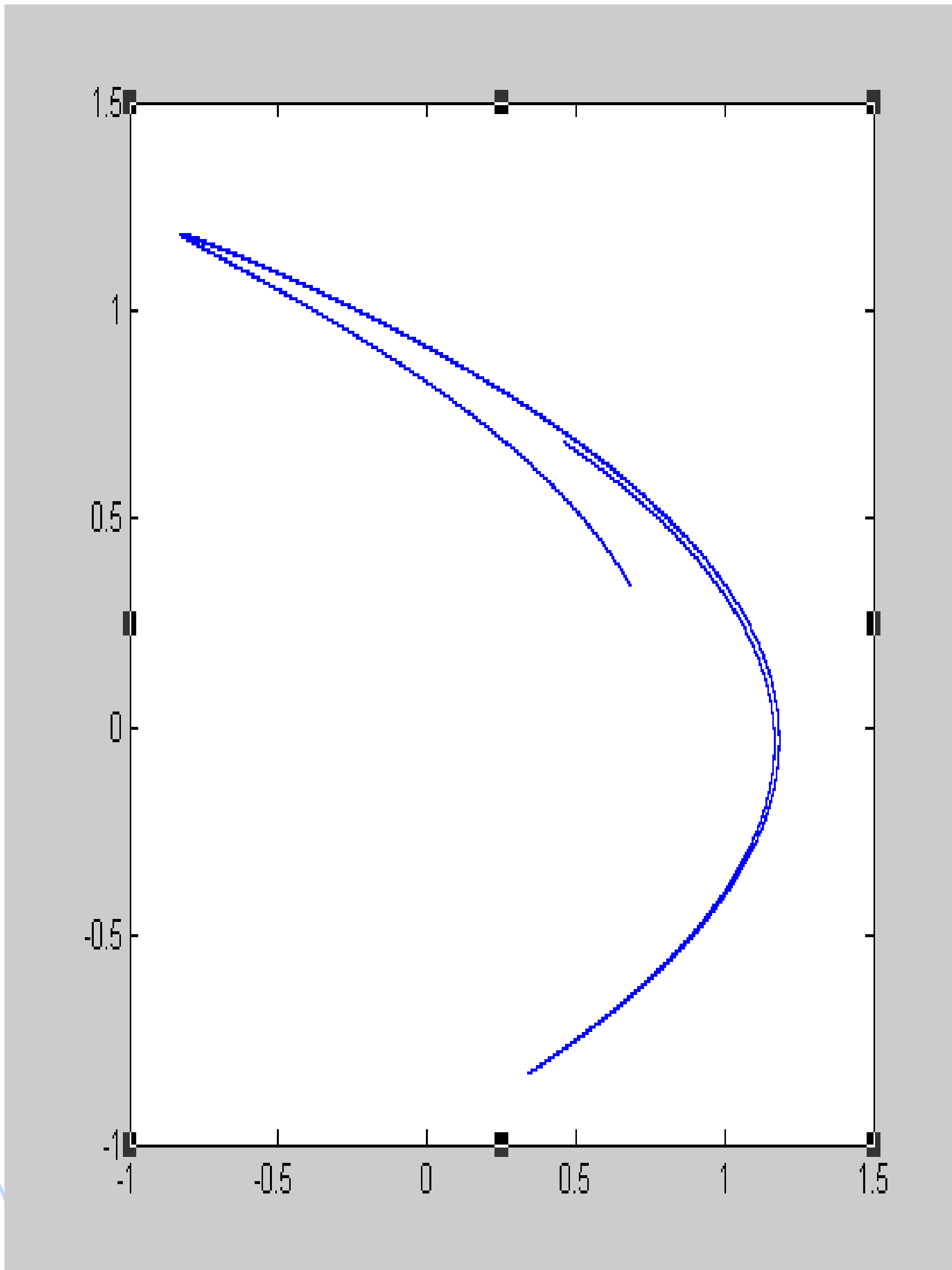
- I use a program called GAIO, which can be used with Matlab. GAIO is useful because it creates a “tree” which separates a given area into boxes at a depth of your own choosing.
- We can then use one of two methods of obtaining a $N(1/\varepsilon)$ for that depth.
  - *GAIO’s Method*
  - *Insert Method*

GAIO’s Method

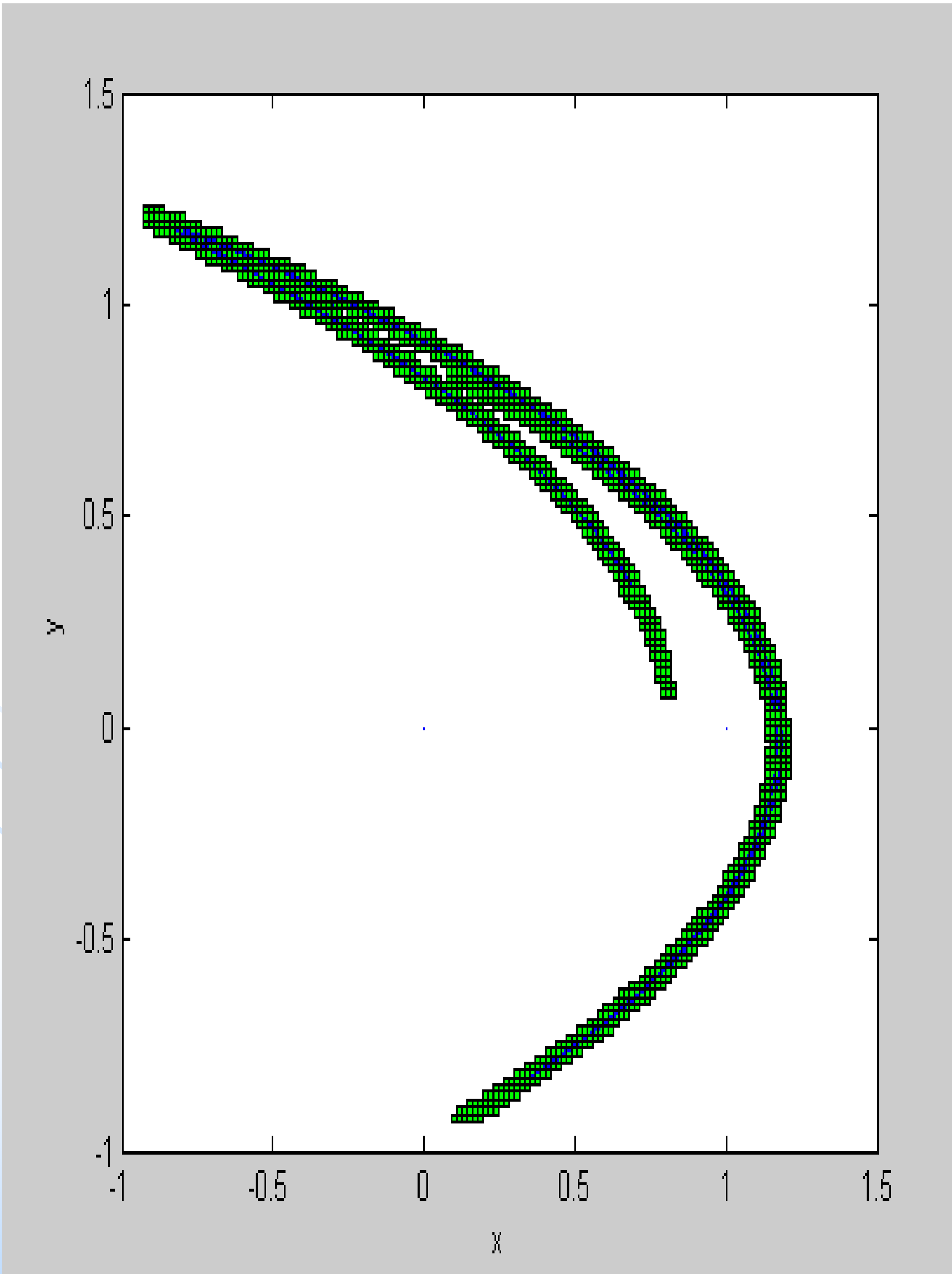
- Pays attention to where boxes are mapped. GAIO then carefully removes those boxes which have no other boxes mapped into it.
- An advantage of this method is that we are almost guaranteed an upper bound for the box-counting dimension.
- A disadvantage is that the process can be slow and can give us a very high estimate of the box-counting dimension.

GAIO's Method

- a = 1.3
- b = 0.2
- Here, I plot the iteration points 100-10,000 with initial point (0,0).



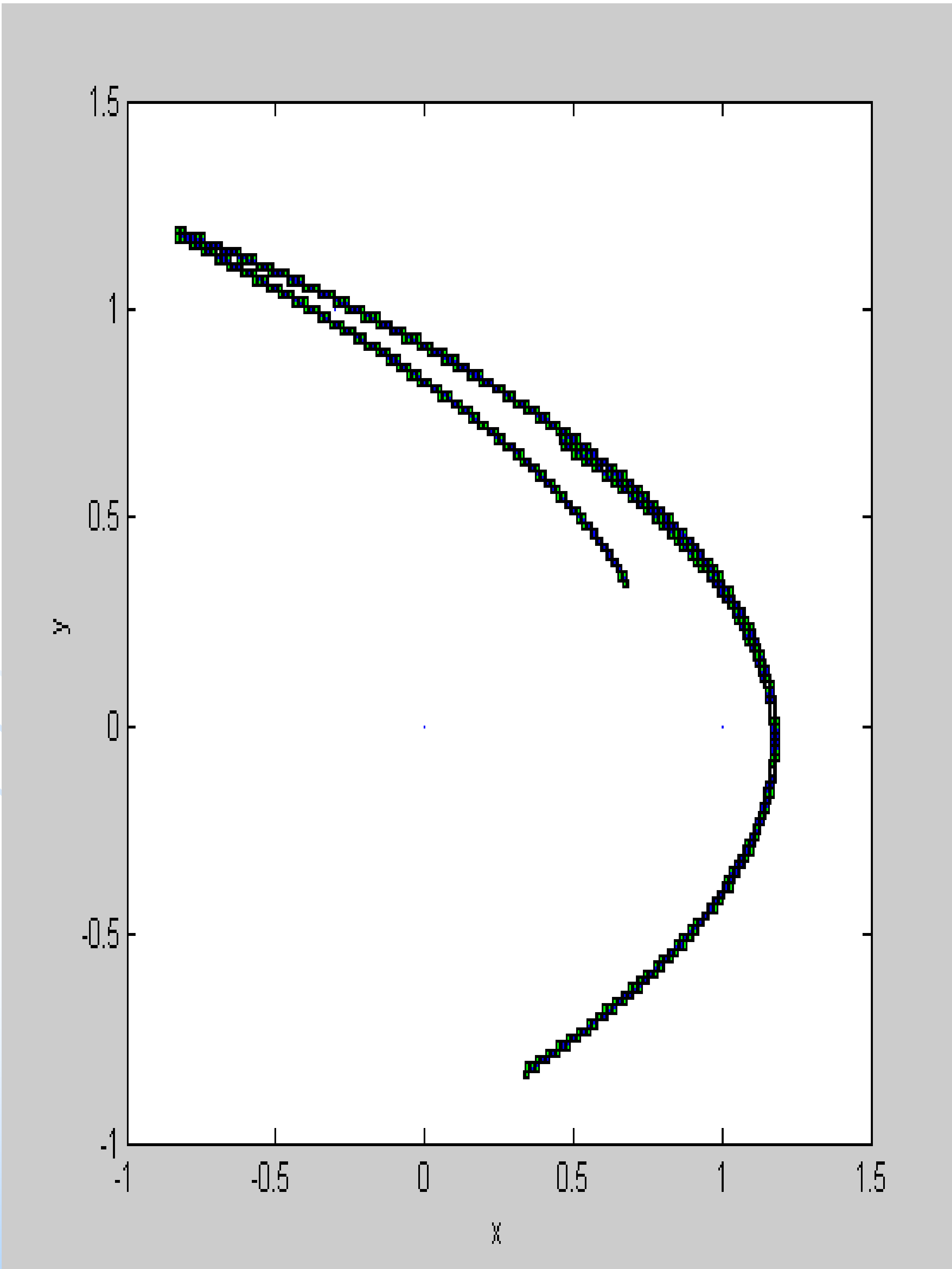
GAIO's Method



Insert Method

- Using this method we only insert boxes which contain iteration points.
- An advantage is that for appropriate values of ε and number of iterations used, this method should give a closer approximation of the box-counting dimension more quickly.
- A disadvantage is that we cannot easily say what these appropriate values are.

Insert Method

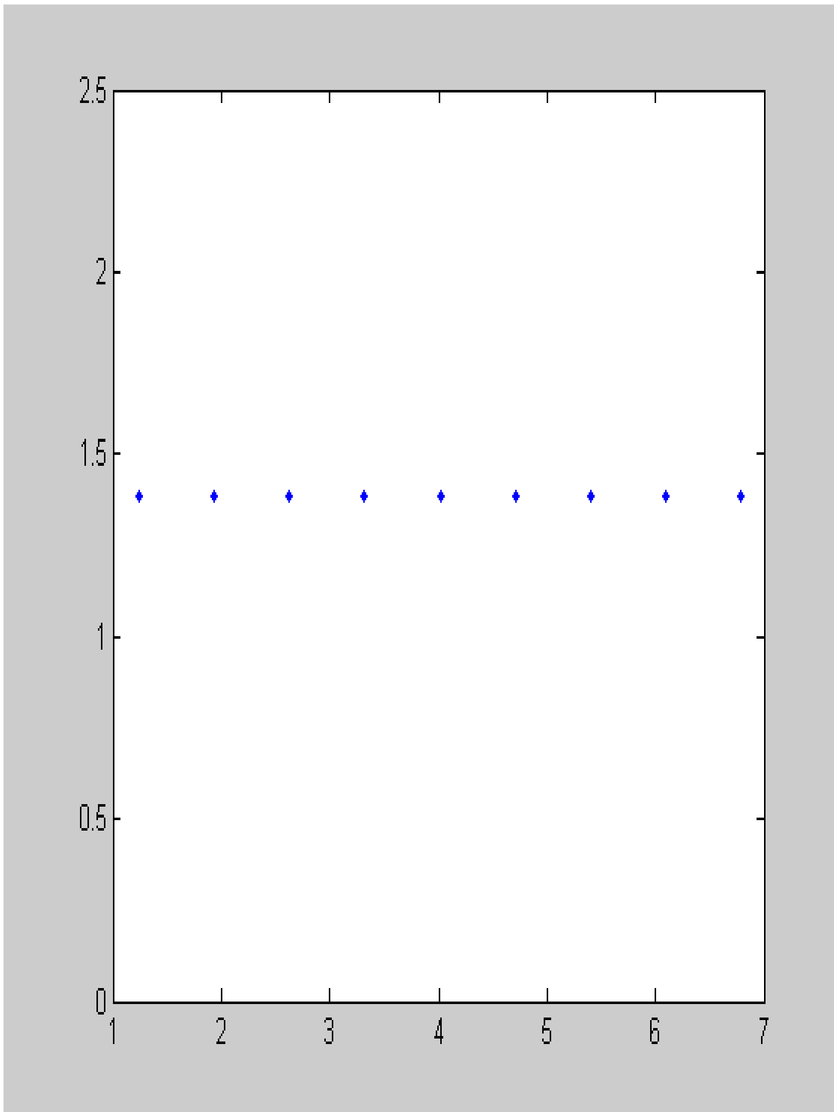


How we compute dimension of attractors

- We can observe the way ln(N)/ln(1/ε) changes for smaller and smaller ε and see where it appears to level off.
- We could also approximate the limit of the slope of ln(1/ε) versus ln(N).
- For both methods, we must keep in mind how limits in the computer's ability can skew results for very small ε.

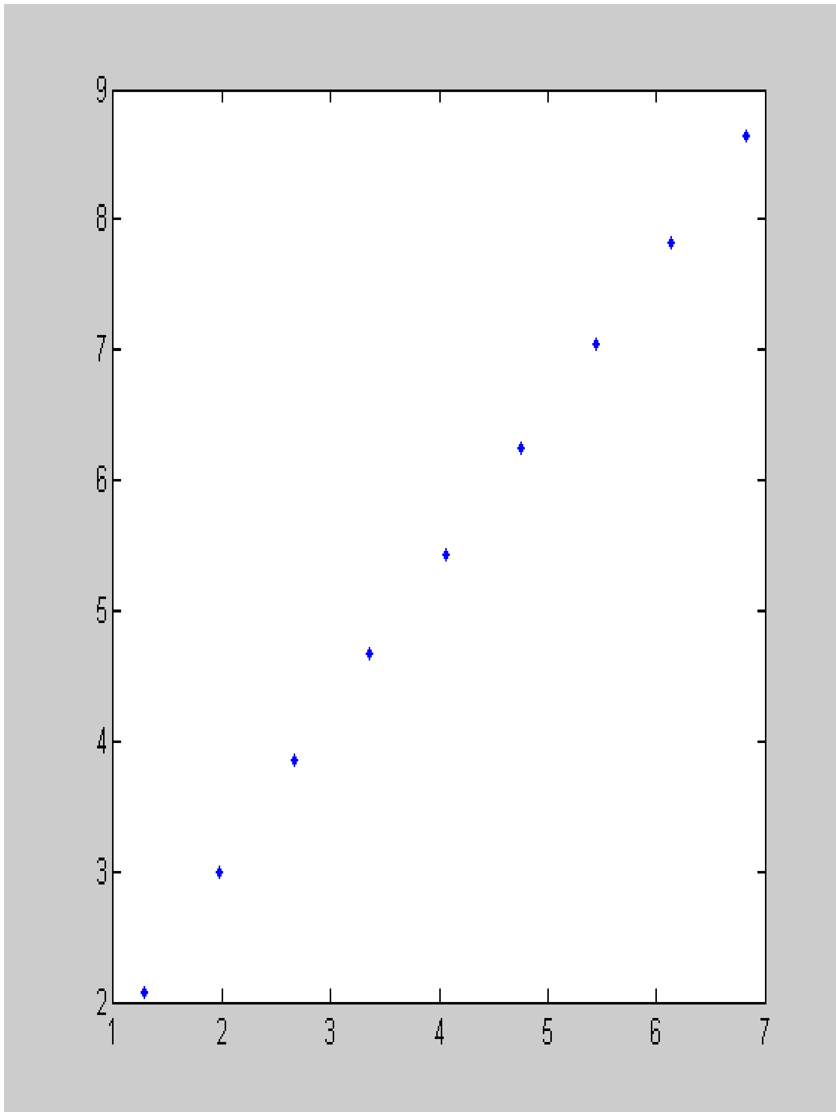
Example

- This is the ln(1/ε) versus ln(N) graph for a set of periodic points. As expected the slope is close to zero.



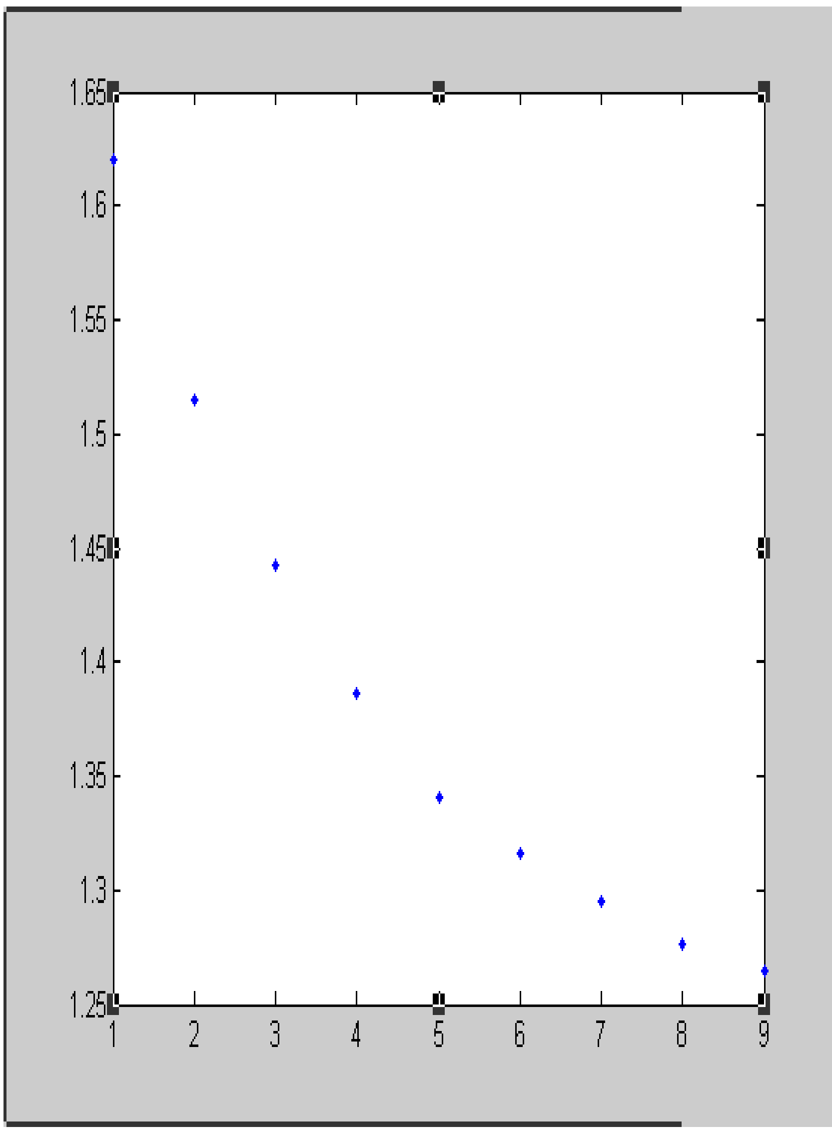
For Hénon Attractor

- This is the ln(1/ε) versus ln(N) graph for the Hénon Attractor we saw earlier. You may notice that the slope is between 1 and 2.



For Hénon Attractor

- This shows the value ln(N)/ln(1/ε) begin to approach what we assume is the true dimension of the attractor asymptotically as ε decreases.



Open Questions

- How to know when we've arrived at the right answer?
- How to know what ε and number of iterations to be used for the insert-method?
- Any better methods of computing box-counting dimension?
- Dimension of other attractors?

Further Open Questions

- How can these methods and ideas help us understand fractals of finite dimension that are subsets of infinite dimensions ?