The Hodge Decomposition Theorem

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Theorem

Let (M, g) be a compact, Riemannian manifold. Then for each $k = 1, ..., n$, the Hilbert space $\Omega^k(M)$ of differential k-forms on M admits an orthonormal decomposition:

$$\Omega^k(M) = \text{im } d \oplus \text{im } \delta \oplus \ker \Delta_k.$$

Where d denotes the exterior derivative, δ denotes the codifferential, and $\Delta_k = d\delta + \delta d$ is the Hodge Laplacian on M.

Corollary

$H^k_{dR}(M) \cong \ker(\Delta_k)$

Motivation

- Topological information (De Rham Cohomology Groups) of M can be inferred by knowing the kernel of the k-Laplacian Δ_k.
- In the case $k = 0$ (and sometimes $k = 1$), the spectrum of Δ_k can be inferred from finite data sampled from M.
- One of the goals for my dissertation is to find a way to infer for $k > 0$.

Overview

- Introduction to differential forms.
- Define common operations $\wedge$, $*$, on differential forms
- Define the *exterior derivative* d as well as the *codifferential* δ.
- Construct the Hodge Laplacian $\Delta = d\delta + \delta d$
- Define the De Rham Cohomology groups $H^k(M)$.

We will see how the Hodge Decomposition Theorem tells us that

$$H^k_{dR}(M) \cong \ker(\Delta_k).$$

Smooth manifolds

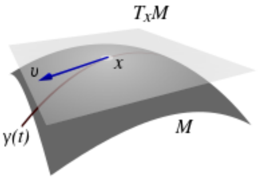
Let M be a smooth manifold of dimension n.

- Hausdorff, second countable topological space that is locally homeomorphic to $\mathbb{R}^n$.
- Equipped with a *smooth structure*, so one can define smooth functions on M.

Examples: $\mathbb{R}^n$, spheres, torus

Tangent Spaces

For each point $p \in M$, one can define the *tangent space at p* denoted T_pM, which is a dimension n vector space.



Given a smooth map $f : M \to N$, we obtain linear maps $df_p : T_pM \to T_{f(p)}N$ on each of the tangent spaces.

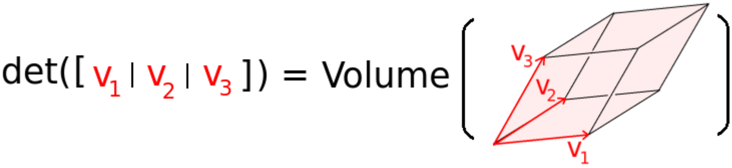
Riemannian Manifolds

- A Riemannian manifold (M, g) is a smooth manifold together with a choice of inner product $\langle \cdot, \cdot \rangle_g$ on each tangent space T_pM.
- This allows us to measure lengths and angles of vectors in each tangent space, as well as the lengths of curves $\gamma : [a, b] \to M$.

Differential Forms

Let M be a smooth manifold of dimension n:

- A differential n-form on M is like choosing a determinant on each tangent space T_pM.



- The determinant is *multilinear*.
- The determinant is *alternating*: If I plug in a set of linearly dependent vectors $v_1, \ldots, v_n$, then the determinant is zero.

Differential Forms

Let M be a smooth manifold of dimension n:

- Formally, a differential n-form ω is a choice of alternating, multilinear map

$$\omega_p : T_pM \times ... \times T_pM \to \mathbb{R}$$

  for each $p \in M$. Meaning that ω_p is zero whenever a linearly dependent set of vectors is inputted. We also require ω_p to vary smoothly across M a function of p.
- Intuitively, this is a way to measure "volume" in each tangent space.

The Riemannian Volume Form

Now let (M, g) be an (orientable) Riemannian manifold of dimension n.

- There is an obvious choice for an n-form.
- If we take an orthonormal set $\{v_1, .., v_n\}$ of vectors in T_pM, the hypercube spanned by the vectors "should" have volume 1.
- There exists a unique n-form on M with this property, called the *Riemannian volume form*, denoted V_g.
- V_g is the differential form that assigns unit volume to unit hypercubes in each T_pM.

k-forms

Now M be a manifold of dimension n and let $k < n$.

- Let $\{e_1, ..., e_n\}$ be a basis for T_pM.
- **Example:** Let $v_1, v_2 \in T_pM$ and Denote $e^1 \wedge e^2(v_1, v_2)$ as the 2-d volume of the square formed by projecting v_1 and v_2 onto the subspace formed by e_1 and e_2.
- $e^1 \wedge e^2$ is a 2-form. (Assuming we construct it across all T_pM.)

k-forms

- For any choice of k unique basis vectors $\{e_{i_1}, ..., e_{i_k}\}$, I can define $e^{i_1} \wedge ... \wedge e^{i_k}$ in the same manner to measure the k dimensional area of vectors in the subspace spanned by $e_{i_1}, ..., e_{i_k}$.
- For technical reasons, we always choose i_k to be strictly increasing.
- Each such $e^{i_1} \wedge ... \wedge e^{i_k}$ can be added and scalar multiplied.

k-forms

- A differential k-form ω can be written as:

$$\omega = \sum_{I=(i_1,\ldots,i_k)} f_I e^I$$

  where $e^I = e^{i_1} \wedge .. \wedge e^{i_k}$ and $f_I : M \to \mathbb{R}$ is smooth. (The f_I represent a choice of linear constants for each fixed T_pM that varies smoothly across M.)
- The set of all differential k-forms on a manifold M is denoted $\Omega^k(M)$.

k-forms

The set $\Omega^k(M)$ is $\binom{n}{k}$ dimensional over $C^\infty M$.

- There are $\binom{n}{k}$ ways to select e^I, and the set of e^I are a pointwise basis for $\Omega^k(M)$.
- Functions in $C^\infty(M)$ denote a way to choose linear constants over each T_pM.
- In the case $k = 0$, we define $\Omega^0(M) = C^\infty(M)$.

The Exterior Derivative

- The Exterior derivative d_k maps k-forms to $k+1$ forms.
- The Fundamental Theorem of Calculus exists on manifolds, and is stated in terms of d (Stokes' Theorem).

$$\int_M d\omega = \int_{\partial M} \omega$$

The Exterior Derivative

- When $k = 0$, a differential form is simply a smooth function $f : M \to \mathbb{R}$. This induces a linear map on tangent spaces:

$$df_p : T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$$

- df_p is 1-multilinear, alternating, and varies smoothly as a function of p. Thus, it is a 1-form!

The Exterior Derivative

- If ω is a k-form and we write:

$$\omega = \sum_{I=(i_1,\ldots,i_k)} f_I e^I$$

then we define $d\omega$ as:

$$d\omega = \sum_{I=(i_1,\ldots,i_k)} df_I \wedge e^I.$$

- We take the 0-form f_I, make it into a 1-form, then "glue" it to e^I with the wedge product, making a $k+1$ form.

The Exterior Derivative

- In practice, we need to know more algebra to compute $d\omega$.
- **Fun Facts:**
  1. If f is a constant function, $df = 0$.
  2. d is linear over $\mathbb{R}$.
  3. $d_{k+1}(d_k\omega) = 0$ for any $\omega \in \Omega^k(M)$.

$$d \circ d = 0.$$

De Rham Cohomology

- $d_{k+1}(d_k\omega) = 0$ implies that $\ker d_{k+1} \subseteq \text{im } d_k$.
- Define
$$H^k_{dR}(M) = \frac{\ker d_{k+1}}{\text{im } d_k}.$$
- $H^k_{dR}(M)$ is a (possibly infinite-dimensional) vector space over $\mathbb{R}$, called the k-th de Rham Cohomology group of M.
- The dimension of $H^k_{dR}(M)$ roughly counts the number of k-dimensional holes in M.

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Corollary

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The Codifferential

The codifferential δ is a map:

$$\delta : \Omega^k(M) \to \Omega^{k-1}(M)$$

defined by:

$$\delta\omega = (-1)^{n(n-k)+1} * d * \omega.$$

- **Fun Facts:**
  1. δ is linear over $\mathbb{R}$
  2. $\delta \circ \delta = 0$

Definition

The k-th Hodge Laplacian $\Delta_k : \Omega^k(M) \to \Omega^k(M)$ is the mapping

$$\Delta_k = \delta d + d\delta.$$

The Hodge Star

Let $e_1, ..., e_n$ be an orthonormal basis for T_pM. Then we see that

$$V_g = e^1 \wedge ... \wedge e^n$$

which measures the *n*-dimensional volume such that

$$V_g(e_1, ..., e_n) = 1.$$

Suppose we take $e^{i_1} \wedge ... \wedge e^{i_k}$. We can determine V_g if we also know $e^{j_1} \wedge ... \wedge e^{j_{n-k}}$ where the j_ℓ's are the indices that are complementary to $i_1, ..., i_k$.

The Hodge Star

The Hodge star $* : \Omega^k(M) \to \Omega^{n-k}(M)$ maps a k-form ω to $*\omega \in \Omega^{n-k}(M)$ such that

$$\omega \wedge *\omega = V_g.$$

This mapping is an isomorphism! so $\Omega^k(M) \cong \Omega^{n-k}(M)$.

The Codifferential

The codifferential $\delta = (-1)^{n(n-k)+1} * d*$ can be explained by the following process:

1. Ignore $(-1)^{n(n-k)+1}$, it's there for algebraic reasons.
2. Imagine a k-form ω as a way to measure k-dimensional subspace of a hypercube.
3. Instead of using k, measure the complementary $n-k$-dimensional volume of the hypercube given by $*\omega$.
4. Take the exterior derivative of $*\omega$ which gives an $n-(k-1)$ dimensional volume.
5. Imagine the volume of the complementary $(k-1)$-dimensional volume which is given by $*d*\omega$.

The Inner Product on $\Omega^k(M)$

We can now define an inner product on $\Omega^k(M)$ by:

$$(\omega, \eta) = \int_M \omega \wedge *\eta$$

- With respect to this inner product, δ is the adjoint to d. For all $\omega \in \Omega^{k-1}(M)$ and ,$\eta \in \Omega^k(M)$

$$(d\omega, \eta) = (\omega, \delta\eta)$$

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- We will show that the mapping

$$\phi : \ker \Delta_k \to H^k_{dR}(M) = \frac{\ker\ d_k}{\text{im}\ d_{k-1}}$$

defined by:

$$\phi(\omega) = [\omega]$$

is bijective.

- Decompose ω:

$$\begin{aligned}\phi(\omega) &= [\omega] \\ &= [\omega_d + \omega_\delta + \omega_\Delta] \\ &= [\omega_d] + [\omega_\delta] + [\omega_\Delta]\end{aligned}$$

Proof of the Corollary

Claim: $[\omega_d] = [0]$

Proof: $\omega_d \in \text{im } d_{k-1}$

Claim: $\omega_\delta = 0$

Proof: Take the exterior derivative.We know $d\omega = 0$.

$$0 = d\omega = d\omega_d + d\omega_\delta + d\omega_\Delta.$$

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Therefore $\omega_\Delta = 0$.

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Therefore $\phi : \ker \Delta_k \to H^k_{dR}(M)$ is an isomorphism.

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In this talk, we have:

- Given an intuitive explanation of all of the terminology necessary to understand the Hodge Decomposition Theorem.
- Explained the relationship between the Hodge Laplacian and De Rham Cohomology.
- Proved that the kernel of the Hodge Laplacian is isomorphic to the De Rham Cohomology Groups of M.

Thank You!

- John M Lee.
  Smooth manifolds.
  In *Introduction to Smooth Manifolds*, pages 1–31. Springer, 2013.

- Steven Rosenberg.
  The Laplacian on a Riemannian manifold: an introduction to analysis on manifolds, volume 31.
  Cambridge University Press, 1997.

- Ulrike Von Luxburg, Mikhail Belkin, and Olivier Bousquet.
  Consistency of spectral clustering.
  The Annals of Statistics, pages 555–586, 2008.

- Frank W Warner.
  Foundations of differentiable manifolds and Lie groups, volume 94.
  Springer Science & Business Media, 2013.