A description of a math circle set of activities around polynomials, especially interpolation.

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I ended up rethinking the Newton idea as a recursive algorithm, which led to another set of topics tied to divisibility of \( p(x) - p(a) \) by \( x - a \) – a wide ranging set of related ideas!
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Didn’t quite get to determinant of it – which is linked to Lagrange form. Never hit on recursive idea – which is Newton (slight variant of usual table).
Was it a success?

- Yes, in that they had fun, were trying things, thinking.

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- No, in terms of getting to the deeper mathematics.
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- Looking for some suggestions – the factored part should have led to deeper insights by factoring the values (How do the factors grow – so if factoring $4x^2 + Bx + C$ either both have $2x$ or one is an $x$ and the other a $4x$, should be easy to see which).

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This leads to Newton interpolation using nested form of polynomial (recursive solution), namely use \( P(x_1) \) at first interpolation point, then factor form of \( P(x) - P(x_1) \) leads to recursion (values at remaining points for adjusted polynomial form).
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Difference quotient comes into play – here automatically nice since polynomials, but an entry into calculus!
Interpolation done recursively

\[ P(x) = P(x_1) + (x - x_1) \cdot P_1(x) \]

where \( P_1 \) takes on values
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\[ P_1(x_j) = \frac{P(x_j) - P(x_1)}{x_j - x_1} \quad \text{for} \quad j = 2, 3, \ldots, n \]

A new interpolation problem, one degree lower, one fewer point.
Interpolation done recursively – next level

\[ P(x) = P(x_1) + (x - x_1) \cdot (P_1(x_2) + (x - x_2) \cdot P_{12}(x)) \]

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Some interesting aspects

- Recursive version breaks the symmetry of the interpolation problem.

What about \( p(x) - p(x-1) \) (finite difference but also table of values idea at any points with separation of size 1)?
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- Usual divided difference table doesn’t show all differences and has no entries like the recursive form.

- Recursive view with one point repeatedly leads to Taylor polynomial for a polynomial.

- What about $p(x) - p(x - 1)$ (finite difference but also table of values idea at any points with separation of size 1)?
Taylor polynomial – key idea

Using recursive division idea with repeated use of one input value yields Taylor polynomial (for a polynomial).

Example: Let $p(x) = x^3$, done in powers of $x-1$ using values at $x=1$, we get:

\[
\begin{align*}
x^3 - 1 &= (x - 1)(x^2 + x + 1) \\
&= (x - 1)(x + 2) = (x - 1)(x + 2) = (x - 1)(3 + (x - 1)(3 + (x - 1)(3)))
\end{align*}
\]

which is binomial / Taylor polynomial expansion of $x^3$ about $x=1$, done purely recursively. Clearly extends to power $x^n$ for positive integer $n$. 
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  x^3 - 1 & = (x - 1)(x^2 + x + 1) \quad \text{note sum of powers} \\
  x^2 + x + 1 - (1 + 1 + 1) & = (x - 1)(x + 1 + 1) = (x - 1)(x + 2)
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x + 2 - (1 + 2) = x - 1
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$$x^3 - 1 = (x - 1)(3 + (x - 1)(3 + (x - 1)))$$

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- Also introduces for algebraic setting the slope of the secant line – in a fun way.

- Since degree decreases, get a proof of usual bad high school description of how to find degree of a polynomial from data values.

- Opens up a lot of alternate questions, especially the following ones.
Can we generalize? Yes – Discrete (difference) calculus

- How about the -1 power? Get a fun recursion that leads to geometric series of powers.

Biggie: how about $f(x)$? Important calculus notion (proving chain rule, e.g.)

How about real numbers? Complex?

Contemplating a session that opens with some examples, then division claim. Then tell them: do something interesting with it.
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- How about real numbers? Complex?
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For the function $f(x) = 1/x$, we find for $x = 1$ expansion (easiest algebra):

$$\frac{1}{x} - 1 = \frac{1 - x}{x} = -(x - 1) \frac{1}{x}$$

which involves the same function we had, so it is a simple recursion to get an infinite series (our favorite, the geometric series!)
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Recursively we find:

\[
\frac{1}{x} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \ldots
\]

as a purely formal algebraic process, perhaps with some motivation.
Case of $1/x^2$

For the function $f(x) = 1/x^2$, we find for $x = 1$ expansion (easiest algebra):

$$\frac{1}{x^2} - 1 = \frac{1 - x^2}{x^2} = -(x - 1) \frac{1 + x}{x^2}$$

which involves both the function we have and also the previous case of $1/x$. 
Case of $1/x^2$

For the function $f(x) = 1/x^2$, we find for $x = 1$ expansion (easiest algebra):

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Recursively we find:

$$\frac{1}{x^2} = 1 - (x - 1) + 2(x - 1)^2 - 3(x - 1)^3 + \ldots$$

as a purely formal algebraic process again.
The last two situations can be extended (at least for the first term) as follows: For the function $f(x) = 1/P(x)$ where $P$ is a polynomial, we find for $x = 1$ expansion (easiest algebra):

$$\frac{1}{P(x)} - \frac{1}{P(1)} = \frac{P(1) - P(x)}{P(x)P(1)} = -(x - 1) \frac{1}{P(1)P_1(x)}$$

which now involves the new polynomial $P_1$ in the denominator.
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\]

which now involves the new polynomial $P_1$ in the denominator. Recursively for the next step we find:

\[
\frac{1}{P_1(x)} - \frac{1}{P_1(1)} = \frac{P_1(1) - P_1(x)}{P_1(x)P_1(1)} = -(x - 1) \frac{1}{P_1(1)P_2(x)}
\]

which can continue for an infinite series expansion.
Another extension is the product rule. Easiest way is to multiply the two Taylor polynomials, but otherwise it goes as follows: For the function \( f(x) = P(x)Q(x) \) where \( P \) and \( Q \) are polynomials, we find for \( x = 1 \) expansion (easiest algebra):

\[
P(x)Q(x) - P(1)Q(1) = (P(x) - P(1))Q(x) + P(1)(Q(x) - Q(1)) \\
= (x - 1)(P_1(x, 1)Q(x) + P(1)Q_1(x, 1))
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Case of $P(x)Q(x)$ – the product rule

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$$P(x)Q(x) - P(1)Q(1) = (P(x) - P(1))Q(x) + P(1)(Q(x) - Q(1))$$

$$= (x - 1)(P_1(x, 1)Q(x) + P(1)Q_1(x, 1))$$

which works but is not symmetric, so one might prefer a more symmetric form at the expense of more algebra.
Case of $N(x)/P(x)$ – the quotient rule

Another extension (at least for the first term, though more in principle) as follows: For the function $f(x) = N(x)/P(x)$ where $N$ and $P$ are polynomials, we find for $x = 1$ expansion (easiest algebra):

$$\frac{N(x)}{P(x)} - \frac{N(1)}{P(1)} = \frac{N(x)P(1) - N(1)P(x)}{P(x)P(1)}$$

which now involves more intricate algebra which we omit. In calculus class, I prefer it as product rule of $N$ and $1/P$. 
Case of $P(Q(x))$ – the chain rule

Another extension is the chain rule. This one goes nicely in factored form (one stage only): For the polynomial function $f(x) = P(Q(x))$

where $P$ and $Q$ are polynomials, we find for $x = 1$ expansion (easiest algebra), with $b = Q(1)$:

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Q(x) - Q(1) = u - b = (x - 1) Q_1(x)
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\[
P(u) - P(b) = (u - b) P_1(u, b)
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P(Q(x)) - P(Q(1)) = (u - b) P_1(u, b) = (x - 1) Q_1(x) P_1(u, b)
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The limits both exist and so this is the discrete version of the chain rule:

\[
\frac{P(Q(x)) - P(Q(1))}{x - 1} = Q_1(x) P_1(Q(x), b)
\]
Case of $x^{\frac{1}{2}}$

For the function $f(x) = \sqrt{x}$, we find for $x = 1$ expansion (easiest algebra):

$$\sqrt{x} - 1 = (x - 1) \frac{1}{\sqrt{x} + 1}$$
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Next step we find:

$$\frac{1}{\sqrt{x} + 1} = \frac{1}{2} + \left( \frac{1}{\sqrt{x} + 1} - \frac{1}{2} \right)$$

which can be evaluated, bit harder algebra. This can also be viewed as inverse function difference question:
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Let $y = f(x) = \sqrt{x}$, we find for $y = 1$ expansion instead (easiest algebra):

$$f_1(x) = \frac{\sqrt{x} - 1}{x - 1} = \frac{y - 1}{y^2 - 1} = \frac{1}{1 + y}$$

which is of course the algebraic form of differencing the inverse function to $g(y) = y^2$. 

R. Sachs (GMU) Polynomials as math circle topics August 2011 18 / 20
Partial fractions

For factored denominators with simple roots, can relate above to partial fraction expansion.

\[ f(x) = \frac{1}{(x-1)(x-a)} \]

For \( x = 1 \) expansion (easiest algebra):

\[ \frac{1}{x-a} - \frac{1}{1-a} = \frac{1}{1-a} - x(x-a) \left( \frac{1}{1-a} \right) = -\frac{(x-1)}{(x-a)(1-a)} \]

which then leads easily to the correct partial fraction version:

\[ \frac{1}{(x-a)(x-1)} = \frac{1}{1-a} \frac{1}{x-1} - \frac{1}{1-a} \frac{1}{x-a} \]

Similarly we can find the expansion for \( x \frac{1}{(x-a)(x-1)} \), so we are done for two roots!

Recursively can peel off singular parts at each pole in fraction. Also works for higher order roots and for repeated quadratic factors (using a suitable generalization).
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Concluding remarks

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- For instance, how to balance case near $x = 1$ versus general location $x = a$; General polynomial versus particular examples; it’s a fine art to extract general cases from specifics – not clear to me how to cultivate that best.

Improvisational math is not easy to lead, but lots of fun to experiment with!

Thanks for any suggestions / advice / feedback from your experiences with these ideas.

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