A description of a math circle set of activities around polynomials, especially interpolation.

Bob Sachs

Department of Mathematical Sciences
George Mason University
Fairfax, Virginia 22030

rsachs@gmu.edu

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- Lagrange version finds dual basis to evaluation at points – not as useful in practice, especially due to updating issue.
- I ended up rethinking the Newton idea as a recursive algorithm, which led to another set of topics tied to divisibility of $p(x) - p(a)$ by $x - a$ – a wide ranging set of related ideas!

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- They were not so great at the game, but had some fun. I hoped two questions spontaneously arise: what is the least number of values needed to pin down the mystery polynomial? How would you actually express it?
- The older kids eventually went to some sytem of linear equations for coefficients of polynomial, ended up with a Vandermonde matrix – couldn't quite convince themselves it was invertible without some help.
- Didn't quite get to determinant of it – which is linked to Lagrange form. Never hit on recursive idea – which is Newton (slight variant of usual table).

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- Looking for some suggestions – the factored part should have led to deeper insights by factoring the values (How do the factors grow – so if factoring $4x^2 + Bx + C$ either both have $2x$ or one is an x and the other a $4x$, should be easy to see which).

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- Schools often have a bad version of this for rational roots, given the polynomial.

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Difference quotient comes into play – here automatically nice since polynomials, but an entry into calculus!

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A new interpolation problem, one degree lower, one fewer point.

Interpolation done recursively – next level

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A newer interpolation problem, one degree lower, one fewer point. Continuing on leads to a nested form solution of the interpolation problem. Naturally there are some good questions to ask: order of points, uniqueness, updating with an extra point/ one degree higher, might introduce idea of linear algebra – various bases for polynomials.

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- Usual divided difference table doesn't show all differences and has no entries like the recursive form.
- Recursive view with one point repeatedly leads to Taylor polynomial for a polynomial.
- What about $p(x) - p(x-1)$ (finite difference but also table of values idea at any points with separation of size 1)?

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which is binomial / Taylor polynomial expansion of x^3 about $x = 1$, done purely recursively. Clearly extends to power x^n for positive integer n.

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Other aspects of interpolation

- Leads naturally to nested multiplication – efficient evaluation in terms of operation count
- Also introduces for algebraic setting the slope of the secant line – in a fun way.
- Since degree decreases, get a proof of usual bad high school description of how to find degree of a polynomial from data values.
- Opens up a lot of alternate questions, especially the following ones.

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- How about real numbers? Complex?
- Contemplating a session that opens with some examples, then division claim. Then tell them: do something interesting with it.

Case of $1/x$

For the function $f(x) = 1/x$, we find for $x = 1$ expansion (easiest algebra):

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which involves the same function we had, so it is a simple recursion to get an infinite series (our favorite, the geometric series!)

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Recursively we find:

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots$$

as a purely formal algebraic process, perhaps with some motivation.

Case of $1/x^2$

For the function $f(x) = 1/x^2$, we find for $x = 1$ expansion (easiest algebra):

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Recursively we find:

$$\frac{1}{x^2} = 1 - (x - 1) + 2(x - 1)^2 - 3(x - 1)^3 + \dots$$

as a purely formal algebraic process again.

Case of $1/P(x)$

The last two situations can be extended (at least for the first term) as follows: For the function $f(x) = 1/P(x)$ where P is a polynomial, we

find for $x = 1$ expansion (easiest algebra):

$$\frac{1}{P(x)} - \frac{1}{P(1)} = \frac{P(1) - P(x)}{P(x)P(1)} = -(x-1)\,\frac{1}{P(1)P_1(x)}$$

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which now involves the new polynomial P_1 in the denominator. Recursively for the next step we find:

$$\begin{aligned} \frac{1}{P_1(x)} - \frac{1}{P_1(1)} &= \frac{P_1(1) - P_1(x)}{P_1(x)P_1(1)} \\ &= -(x-1)\,\frac{1}{P_1(1)P_{11}(x)} \end{aligned}$$

which can continue for an infinite series expansion.

Case of $P(x)Q(x)$ – the product rule

Another extension is the product rule. Easiest way is to multiply the two Taylor polynomials, but otherwise it goes as follows: For the

function $f(x) = P(x)Q(x)$ where P and Q are polynomials, we find for $x = 1$ expansion (easiest algebra):

$$\begin{aligned} P(x)Q(x) - P(1)Q(1) &= (P(x) - P(1))Q(x) + P(1)(Q(x) - Q(1)) \\ &= (x-1)\big(P_1(x,1)Q(x) + P(1)Q_1(x,1)\big) \end{aligned}$$

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which works but is not symmetric, so one might prefer a more symmetric form at the expense of more algebra.

Case of $N(x)/P(x)$ – the quotient rule

Another extension (at least for the first term, though more in principle) as follows: For the function $f(x) = N(x)/P(x)$ where N and P are

polynomials, we find for $x = 1$ expansion (easiest algebra):

$$\frac{N(x)}{P(x)} - \frac{N(1)}{P(1)} = \frac{N(x)P(1) - N(1)P(x)}{P(x)P(1)}$$

which now involves more intricate algebra which we omit. In calculus class, I prefer it as product rule of N and $1/P$.

Case of $P(Q(x))$ – the chain rule

Another extension is the chain rule. This one goes nicely in factored form (one stage only): For the polynomial function $f(x) = P(Q(x))$

where P and Q are polynomials, we find for $x = 1$ expansion (easiest algebra), with $b = Q(1)$:

$$\begin{aligned}
Q(x) - Q(1) &= u - b = (x-1)\,Q_1(x) \\
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The limits both exist and so this is the discrete version of the chain rule:

$$\frac{P(Q(x)) - P(Q(1))}{x-1} = Q_1(x)\,P_1(Q(x),b)$$

Case of $x^{\frac{1}{2}}$

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which can be evaluated, bit harder algebra. This can also be viewed as inverse function difference question:

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Let $y = f(x) = \sqrt{x}$, we find for $y = 1$ expansion instead(easiest algebra):

$$f_1(x) = \frac{\sqrt{x}-1}{x-1} = \frac{y-1}{y^2-1} = \frac{1}{1+y}$$

which is of course the algebraic form of differencing the inverse function to $g(y) = y^2$.

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Similarly we can find the expansion for $\frac{x}{(x-1)(x-a)}$, so we are done for two roots!
Recursively can peel off singular parts at each pole in fraction. Also works for higher order roots and for repeated quadratic factors (using a suitable generalization).

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- Improvisational math is not easy to lead, but lots of fun to experiment with!
- Thanks for any suggestions / advice / feedback from your experiences with these ideas