MATH 675 — Homework 3 Comments / Solutions Prof Sacks

① Section 12, Problem 6 — Monotonicity leads to an open covering

Idea — Seeking $N(\varepsilon)$ so that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in K$ and all $n \geq N(\varepsilon)$

Most of you created sets $O_{n,\varepsilon} = \{ x \in K : |f_n(x) - f(x)| < \varepsilon \}$ and noted

$O_{n,\varepsilon}$ is open, $\bigcup_n O_{n,\varepsilon} = K$ so finite subcovering exists

② Section 13, Problem 6 — A tricky one! Some student ideas are described briefly (alphabetical order of student)

Ashley L.: Use Theorem 3 recursively — for any $x \in L$

$x = \alpha_1 x_1 + y_1$, $f_1(x_1) \neq 0$, $f_1(y_1) = 0$. Modify x_1 if needed

So $f_2(y_1) \neq 0$ and split it so $x = \alpha_1 x_1 + \alpha_2 x_2 + y_2$ where $y_2 \in L_{f_1} \cap L_{f_2}$

Continue — get $x = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n + y_n$; $y_n \in L_{f_1} \cap L_{f_2} \cdots \cap L_{f_n}$

Use these to find $a_1, a_2, \dots, a_n$ — but vague on details

Patrick O.: Similar, idea $y_{k-1} = \alpha_k x_k + y_k$ and then continue

using $N_k = \bigcap_{j=1}^{k} L_{f_j}$ — more details but need a_k constant too.

Amy S. Pick $x_0 \in \bigcap_{j=1}^{n} (L \setminus L_{f_j}) \cap (L \setminus L_f)$

write $x = \alpha x_0 + y = \alpha_1 x_0 + y_1 = \alpha_2 x_0 + y_2 = \dots = \alpha_n x_0 + y_n$

Now $f(x - a_0 x_0) + \sum_{i=1}^{n} f_i(x - a_i x_0) = 0$

or $f(x) - a f(x_0) + \sum_{i=1}^{n} f_i(x) - \sum_{i=1}^{n} a_i f_i(x_0) = 0$

Now let $x \in L_{f_j}$ for all j (which implies by hypothesis L_f too)

So $- a f(x_0) - \sum_{i=1}^{n} a_i f_i(x_0) = 0$. (lot of fudging now - constants a_i?)

~~Nana W. Recursive lists again - like Ashley, Patrick~~ All $\cancel{x}$ - in some, all, none?

~~My notation By hypothesis $f = 0$ on $L_{f_1} \cap L_{f_2} \cdots \cap L_{f_n}$~~

Assume the "tower" has all proper extensions

$$L \supset L_{f_1} \supset (L_{f_1} \cap L_{f_2}) \supset \cdots \supset (L_{f_1} \cap L_{f_2} \cdots \cap L_{f_n})$$

Goal For all x, there are constants $a_1, \ldots, a_n$ so that

$f(x) = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x)$. Use Theorem 3

repeatedly, starting from $L_{f_1} \cap L_{f_2} \cdots \cap L_{f_n}$ where any a_j works.

From Theorem 3, can find a_n so that $f(x) - a_n f_n(x) \equiv 0$

for all $x \in L_{f_1} \cap L_{f_2} \cdots \cap L_{f_{n-1}}$. Applying theorem 3 again to

$f - a_n f_n$, f_{n-1} to conclude there is a constant a_{n-1} so that

$f - a_n f_n - a_{n-1} f_{n-1}$ is identically 0 on $L_{f_1} \cap L_{f_2} \cdots L_{f_{n-2}}$

Continue to reach L after n steps.

Alternative - use ideas above with $x_1, x_2, \ldots, x_n$ - use these to find $a_1, \ldots, a_n$ then show the difference vanishes on L.

3. Mostly OK — need special y to show $x \notin +y$ doesn't stay inside for $|t|>0$.

4. As discussed in class: Need some language to cross from Cauchy sequence of equivalence classes to a Cauchy sequence of $\{x_n\}$ where $[x_n + M]$ are the equivalence classes,

5. Mostly fine — were limit is in the space.

6. Mostly fine

7. Some of you were too glib — limits here are now in $\mathbb{R}^+$ so equivalence classes of Cauchy sequences, so some discussion is needed [it's not that hard], as in

$\{x_n\} \sim \{x_n'\}$, $\{y_n\} \sim \{y_n'\}$ then $\lim_{n\to\infty}(x_n, y_n) = \lim_{n\to\infty}(x_n', y_n')$

Your writing is getting better — keep improving!