MATH 675 – HMWK 2 – Remarks / Solutions   Prof SACHS

(1) Section 7, Problem 2 – A few of you confused level of sequences [sequences of sequences] – also a notation issue

Use $X^{(j)} = (x_1^{(j)}, x_2^{(j)}, x_3^{(j)}, \ldots)$ Cauchy if $\rho(X^{(i)}, X^{(j)}) < \varepsilon$ for all $i,j \geq N(\varepsilon)$

Now each component k has entries with $|x_k^{(i)} - x_k^{(j)}| < \varepsilon$ for all $i,j \geq N$ so $x_k^{(i)} \to x_k^*$ as $i \to +\infty$. [Cauchy in components]

Then definitions show $(x_1^*, x_2^*, \ldots, x_k^*, \ldots) \in m$ so m is complete.

(2) This uses definitions – not complicated → OK for all.

(3) $x - \lambda F(x) = f(x)$ want to use Fixed Point [Banach contracting mapping] on $[a,b]$. Need f maps $[a,b]$ to $[a,b]$ and

$|f(x_1) - f(x_2)| \leq \alpha |x_1 - x_2|$ for all $x_1, x_2 \in [a,b]$

By calculus, suffices to have $|f'(c)| \leq \alpha < 1$ for all $c \in [a,b]$ for contraction; might need more for $f(x) \in [a,b]$ – check. (often missing in papers)

Contraction ($\lambda > 0$ since $f'(x) = 1 - \lambda F'(x)$)

$-\alpha \leq 1 - \lambda F'(x) \leq \alpha$, $0 < \alpha < 1$.

This becomes $\lambda F'(x) \leq 1+\alpha < 2$

So using $F'(x) \geq K_1$ we get $1/F'(x) \leq 1/K_1$ so $\lambda < \frac{2}{K_1}$

Also $F(a)<0$, $F(b)>0$ so ~~$f(a) = a - \lambda F(a) > a$~~
$\lambda > 0$

need $f(a) < b$, so $a - \lambda F(a) < b$ or $-\lambda F(a) < b-a$

$\lambda < \frac{b-a}{-F(a)}$ too. Also $f(b) = b - \lambda F(b) < b$ but require

$f(b) = b - \lambda F(b) > a$ so $\frac{b-a}{F(b)} > \lambda$ – Pick min of these three

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(4) Section 8, Problem 7 — Similar to linear example – use

$$\left| \int_a^b K(x,y;f_1(y))\,dy - \int_a^b K(x,y;f_2(y))\,dy \right|$$

$$\leq \int_a^b |M(f_1(y)-f_2(y))|\,dy = \underset{\text{length}}{(b-a)} \cdot M \cdot \max_y |f_1(y)-f_2(y)|$$

(5) Went pretty well on most papers; C_{xy} closed in M_{xy} since uniform limit of continuous functions is continuous.