HOMEWORK 1 SOLUTION
Math 675 – Fall 2012 Prof Sachs

(1) Section 5 Problem 1 – uses triangle inequality – seems OK to all
<u>BUT</u> watch out <u>subtracting</u> inequalities

(2) Section 5, Problem 3 – Expand the third term to get the result:

$$\frac{1}{2}\int_a^b\int_a^b\left[x(s)y(t)-y(s)x(t)\right]^2 ds\,dt = \frac{1}{2}\int_a^b\int_a^b\left[x^2(s)\,y^2(t) - 2x(s)y(s)x(t)y(t) + x^2(t)y^2(s)\right] ds\,dt$$

[Note s, t are "loop" variables and products of integrals ensue]

$$= \frac{1}{2}\|x\|^2\|y\|^2 - \left[\int_a^b x(s)y(s)\,ds\right]^2 + \frac{1}{2}\|x\|^2\|y\|^2$$

where $\|x\|^2 = \int_a^b x^2(t)\,dt = \int_a^b x^2(s)\,ds$ etc. Using this we have

$$\left(\int_a^b x(t)\,y(t)\,dt\right)^2 = \|x\|^2\|y\|^2 - \frac{1}{2}\left[2\|x\|^2\|y\| - 2\left(\int_a^b x(t)y(t)\,dt\right)^2\right]$$

$$= \left(\int_a^b x(t)y(t)\,dt\right)^2 \text{ so identity holds.}$$

Therefore Schwarz's inequality follows with equality only when

$$\int_a^b\int_a^b\left[x^2(s)\,y^2(t) - x^2(t)\,y^2(s)\right] ds\,dt = 0$$

(3) Section 5, Problem 6 – Hölder's inequality from $ab \le \frac{a^p}{p} + \frac{b^q}{q}$

If $\int_a^b |x(t)|^p\,dt = 0$ or $\int_a^b |y(t)|^q\,dt = 0$ for continuous x, y

we get $0 \le 0$ – nothing to prove! So assume <u>non-zero</u>

Consider $\tilde{x}(t) = \dfrac{x(t)}{\left(\int_a^b |x(s)|^p ds\right)^{1/p}}$ and $\tilde{y}(t) = \dfrac{y(t)}{\left[\int_a^b |y(s)|^q ds\right]^{1/q}}$

So $\int_a^b |\tilde{x}(t)|^p dt = \int_a^b |\tilde{y}(t)|^q dt = 1$. Then using (19) we have

$|\tilde{x}(t)\,\tilde{y}(t)| \le \frac{1}{p}|\tilde{x}(t)|^p + \frac{1}{q}|\tilde{y}(t)|^q$ and therefore integrating and a simple fact gives the desired result!

$$\left|\int_a^b \tilde{x}(t)\tilde{y}(t)\,dt\right| \le \int_a^b |\tilde{x}(t)\,\tilde{y}(t)|\,dt \le \int_a^b \frac{1}{p}|\tilde{x}(t)|^p dt + \int_a^b \frac{1}{q}|\tilde{y}(t)|^q dt = 1$$

unrolling this

$$\left|\int_a^b x(t)y(t)\,dt\right| \le \left[\int_a^b |x(t)|^p dt\right]^{1/p}\left[\int_a^b |y(t)|^q dt\right]^{1/q} \checkmark$$

(4) Section 5, Problem 7 – (Minkowski's inequality for LP)

Following discrete version (Example 10, p41) is pretty direct

$p=1$ – easy $(q=\infty)$ so assume $p>1, q>1$

Use $\int_a^b |x(t)+y(t)|^p dt \le \int_a^b (|x(t)|+|y(t)|)^p dt$

$$= \int_a^b (|x(t)|+|y(t)|)^{p-1}|x(t)|\,dt + \int_a^b (|x(t)|+|y(t)|)^{p-1}|y(t)|\,dt$$

Now use Hölder's inequality as in text example to deduce

$$\left[\int_a^b (|x(t)|+|y(t)|)^p\,dt\right]^{1-1/q} \le \left(\int_a^b |x(t)|^p dt\right)^{1/p} + \left[\int_a^b |y(t)|^p dt\right]^{1/p}$$

and note $1-1/q = 1/p$ so Minkowski's inequality (triangle inequality in L^p) holds.

(5) Problem on stereographic projection – let (X, Y, Z) be point on sphere from $(x,y,0)$ along segment towards $(0,0,1)$. Some algebra yields the mapping

$$(x,y,0) \leftrightarrow \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{(x^2+y^2)-1}{x^2+y^2+1}\right) \equiv (X, Y, Z)$$

So new distance $\left[\rho\left((x_1,y_1,0),(x_2,y_2,0)\right)\right]^2 = (X_1-X_2)^2+(Y_1-Y_2)^2+(Z_1-Z_2)^2$

Writing $x_1^2+y_1^2=r_1^2$, $x_2^2+y_2^2=r_2^2$ we get after some algebra (recall pts are on sphere)

$$\left[\rho\left((x_1,y_1,0),(x_2,y_2,0)\right)\right]^2 = X_1^2-2X_1X_2+X_2^2+Y_1^2-2Y_1Y_2+Y_2^2$$
$$+Z_1^2-2Z_1Z_2+Z_2^2 = 2-2(X_1X_2+Y_1Y_2+Z_1Z_2)$$
$$= 2-2\,\frac{x_1x_2+y_1y_2+(r_1^2-1)(r_2^2-1)}{(r_1^2+1)(r_2^2+1)}$$ which simplifies further if needed

$$= \frac{2\left[(x_1-x_2)^2+(y_1-y_2)^2\right]}{(r_1^2+1)(r_2^2+1)}$$ (note that Euclidean distance gets "rescaled" by the denominator and the factor 2)

This is a metric because the distance in $\mathbb{R}^3$ is, which gets "inherited" by the unit sphere in $\mathbb{R}^3$.

Also as $(X_2, Y_2, Z_2) \to (0,0,1)$ we get

$$\rho((x_1,y_1,0), \text{"}\infty\text{"}) = \sqrt{(X_1-0)^2+(Y_1-0)^2+(Z_1-1)^2}$$

algebra: $\frac{x^2+y^2-1-(x^2+y^2+1)}{x^2+y^2+1}$

$$= \sqrt{\frac{4x^2+4y^2+4}{(x^2+y^2+1)^2}}$$

$$= \frac{2}{\sqrt{x^2+y^2+1}}$$

Hope this was interesting!

Section 6, problem 2

(6) This uses definitions! If x is a contact point, then either every neighborhood of x contains infinitely many points or some ε_0-neighborhood N of x has only finitely many points of M. In the latter case we let $\varepsilon_* = \min_{x_j \in N \setminus \{x\}} \rho(x_j, x)$ and then for any $\varepsilon < \varepsilon_*$, the ε-ball around x only contains x. Then $x \in M$ is an isolated point.

(7) Section 6, problem 3

Using hint and triangle inequality – straightforward

$|\rho(x_n,y_n) - \rho(x,y)| \le \rho(x_n,x) + \rho(y_n,y)$ use $\varepsilon/2$ or limits

<u>COMMENTS:</u> (a) Watch inequalities and signs

(b) Mostly really nicely done

(c) Exploit symmetry in cases if possible