MATH 414 — HOMEWORK 1 SOLUTION   Spring 2007, Prof. SACHS

1.5.4

(a) $\begin{vmatrix} \lambda-3 & -4 \\ -4 & \lambda+3 \end{vmatrix} = \lambda^2 - 9 - 16 = \lambda^2 - 25$ so $\lambda = \pm 5$

$\lambda = 5$: $\begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for $\lambda = 5$

$\lambda = -5$: $\begin{pmatrix} -8 & -4 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for $\lambda = -5$

Orthogonal eigenvectors ✓   trace $A_1 = 0 = (\lambda_1 + \lambda_2)$ ✓

det $A_1 = -9 - 16 = -25 = \lambda_1\lambda_2$ ✓

(b) $\begin{vmatrix} \lambda-1 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-1 \end{vmatrix} = (\lambda-1)^2(\lambda-2) - (\lambda-1) - (\lambda-1) = (\lambda-1)\left[(\lambda-1)(\lambda-2) - 2\right]$

$= (\lambda-1)\,\lambda\cdot(\lambda-3)$

eigenvectors

$\lambda = 1$: $\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow b = 0,\ a + c = 0 \Rightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda = 3$: $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ works

$\lambda = 0$: $\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow a = b = c$: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

orthogonality ✓   trace: $4 = 1 + 3 + 0$ ✓

det: $1 - 1 = 0 = 0\cdot 1\cdot 3$ ✓

$\boxed{1.5.5}$ From (a) above, $u(t) = c_1\begin{pmatrix}2\\1\end{pmatrix}e^{5t} + c_2\begin{pmatrix}1\\-2\end{pmatrix}e^{-5t}$

At $t=0$: $\left.\begin{aligned}2c_1+c_2&=0\\c_1-2c_2&=6\end{aligned}\right\}$ $c_1 = 6/5$, $c_2 = -\frac{12}{5}$

Answer $\left(\frac{6}{5}\right)\begin{pmatrix}2\\1\end{pmatrix}e^{5t} - \frac{12}{5}\begin{pmatrix}1\\-2\end{pmatrix}e^{-5t}$ ok

or $\vec{u} = \begin{pmatrix}12/5\left(e^{5t}-e^{-5t}\right)\\ 6/5\, e^{5t} + \frac{24}{5}e^{-5t}\end{pmatrix}$

$\boxed{1.5.6}$ $\frac{d^2\vec{u}}{dt^2} + A_2\vec{u} = 0$, $\vec{u}(0) = \begin{bmatrix}2\\-1\\-1\end{bmatrix}$ and $\vec{u}'(0) = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Sol$^{\underline{n}}$ $\left[a_1\cos(\sqrt{1}\,t) + b_1\sin(\sqrt{1}\cdot t)\right]\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$+ \left[a_2\cos(\sqrt{3}\,t) + b_2\sin(\sqrt{3}\cdot t)\right]\begin{pmatrix}1\\-2\\1\end{pmatrix}$

$+ \left(a_3 + b_3 t\right)\begin{pmatrix}1\\1\\1\end{pmatrix}$

At $t=0$: $\vec{u}(0) = a_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + a_2\begin{pmatrix}1\\-2\\1\end{pmatrix} + a_3\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix}$

Using orthogonality(!!) $a_1 = (1\ 0\ -1)\begin{pmatrix}2\\-1\\-1\end{pmatrix} \Big/ \underbrace{2}_{\vec{e}_1\cdot\vec{e}_1} = \frac{3}{2}$ ✓

$a_2 = (1\ -2\ 1)\begin{pmatrix}2\\-1\\-1\end{pmatrix} \Big/ (1+4+1) = \frac{1}{2}$ ✓

$a_3 = (1\ 1\ 1)\begin{pmatrix}2\\-1\\-1\end{pmatrix} \Big/ 3 = 0$ ✓

1.5.6 continued

As for $\vec{u}'(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, by linear indep. we get $b_1 = b_2 = b_3 = 0$

$$\boxed{\text{Sol} = \tfrac{3}{2}\cos t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \tfrac{1}{2}\cos(\sqrt{3}t)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$$

$\boxed{1.5.12}$ If $\det(A - \lambda I) = \prod_{j=1}^{n} (\lambda_j - \lambda)$

then for $\lambda = 0$ we get

$$\det A = \prod_{j=1}^{n} \lambda_j \quad ✓$$

$\boxed{1.5.13}$ From factor form, coeff. of $(-\lambda)^{n-1}$ has one choice of λ_j in product, for each j, added

so coeff of $(-\lambda)^{n-1}$ in $\det(A - \lambda I) = \sum_{j=1}^{n} \lambda_j$

For $\det(A - \lambda I)$, need at least $n-2$ diagonal terms (if not, degree in λ is too small) so we get for example

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} \cdot (a_{22} - \lambda)(a_{33} - \lambda) \cdots (a_{nn} - \lambda)$$ with all other minors of too low a degree

Now 2×2 is degree 2 in λ, so this becomes

$[(a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12}](a_{22} - \lambda) \cdots (a_{nn} - \lambda)$ and the second term (off-diagonal), $-a_{21}a_{12}$, gives no λ, so degree $\leq n-2$ there. This leaves

$$\text{coeff of } (-\lambda)^{n-1} \text{ in } \det(A-\lambda I) = \text{coeff. of } (-\lambda)^{n-1} \text{ in } \prod_{j=1}^{n}(a_{jj}-\lambda)$$

which as in earlier statement, becomes all choices of $n-1$ terms using $-\lambda$, one using a_{jj} $\to$ ie trace and trace(A)

$$= \sum_{j=1}^{n} \lambda_j$$

Algebra fans note: coeff of $(-\lambda)^{n-k}$ in factorization

$(\lambda_1-\lambda)\cdots(\lambda_n-\lambda)$ uses $n-k$ factors $-\lambda$ so k factors λ_j

- get $\displaystyle\sum_{i_1<i_2<\cdots<i_k} \underbrace{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}}_{k \text{ distinct terms in product}}$