MATH 413 — Fall 2006

HOMEWORK 8 SOLUTION

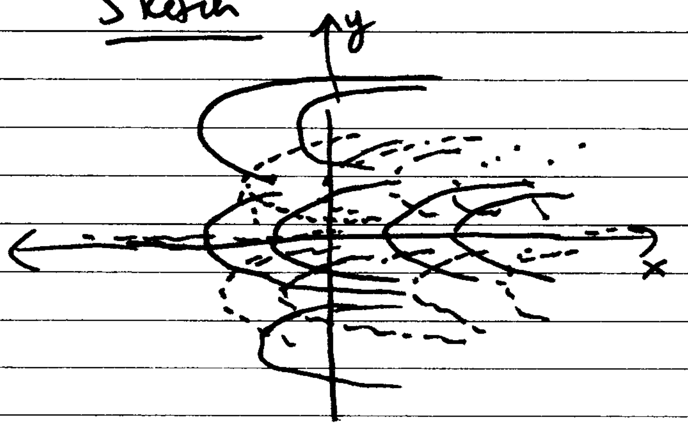
3.3.2 $u = e^x \cos y$, $s = e^x \sin y$

$\frac{\partial^2 u}{\partial x^2} = e^x \cos y$, $\frac{\partial^2 u}{\partial y^2} = -e^x \cos y$ so u is harmonic

Similarly $\frac{\partial^2 s}{\partial x^2} = e^x \sin y$, $\frac{\partial^2 s}{\partial y^2} = -e^x \sin y$ so s is harmonic.

Also $\vec{\nabla} u \cdot \vec{\nabla} s = 0$ everywhere, so level sets are orthogonal.

Sketch



$e^x \cos y$ ——

$e^x \sin y$ - - - - (shift by $\pi/2$ in y)

3.3.3 Across cut: (From A side to F side) get $d_3 - d_9 + d_8 - d_5$ + for

Sum from A, B, C, D : $\underbrace{(d_1 - d_6)}_{A} + \underbrace{(-d_1 + d_2 + d_7 - d_9)}_{B}$

$+ \underbrace{(-d_5 + d_6 - d_7 + d_8)}_{C} + \underbrace{(-d_2 + d_3)}_{D}$

Consists of edges connecting (twice, opp. signs) which cancel from sum and those crossing. CUT.

$\boxed{3.3.10}$ If $v_1 = -y$, $v_2 = 0$ then $\iint \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\, dx\,dy = \iint 1 \cdot dx\,dy$
$= \text{area}$

So area $= \oint -y\,dx$ - ellipse: $x = a\cos t$, $y = b\sin t$ so line integral is $\int_0^{2\pi} -b\sin t(-a\sin t)\,dt = ab\int_0^{2\pi}\sin^2 t\,dt = ab\cdot\frac{1}{2}2\pi$
$= \pi ab$

[Do $\sin^2 t$ or recall $\sin^2 t + \cos^2 t = 1$, $\cos^2 t - \sin^2 t = \cos(2t) \leftarrow$ integrates to 0
or shift to know $\sin^2 t$ over period averages $\frac{1}{2}$

$\boxed{3.3.11}$ $v_1 = y^2$, $v_2 = x^2$ curl is $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 2x - 2y$

$v_1 = y^2$, $v_2 = 2xy$, curl is $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 2y - 2y = 0$

If $\frac{\partial u}{\partial x} = y^2$, $\frac{\partial u}{\partial y} = 2xy$ we get $u = xy^2 + g(y)$ from x-part
and then $g(y) = C$ since $\frac{\partial u}{\partial y} = 2xy + g'(y)$
$= 2xy$

Answer: $u = xy^2 + C$

$\boxed{3.3.17}$ $u = r\cos\theta + \frac{1}{r}\cos\theta \Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$

$= 0\cdot\cos\theta + \frac{2}{r^3}\cos\theta + \frac{1}{r}\left[\cos\theta - \frac{1}{r^2}\cos\theta\right]$
$+ \frac{1}{r^2}\left[-r\cos\theta - \frac{1}{r}\cos\theta\right]$
$= \left(\frac{1}{r} - \frac{1}{r}\right)\cos\theta + \left(\frac{2}{r^3} - \frac{1}{r^3} - \frac{1}{r^3}\right)\cos\theta$
$= 0$ ✓

$\vec{v} = \vec{\nabla} u = (u_x, u_y)$

[easier in polar coords., see below]

$u = x + \frac{x}{x^2+y^2}$

$\frac{\partial u}{\partial x} = 1 + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$ and $\vec{n} = (x, y)$ on unit circle $\rightarrow$ get the flow on body is $x\left[1+\frac{1}{x^2+y^2}-\frac{2x^2}{(x^2+y^2)^2}\right]-\frac{2xy^2}{(x^2+y^2)^2}$

$= x + \frac{x}{x^2+y^2} - \frac{2x}{(x^2+y^2)^2}(x^2+y^2)$ But $x^2+y^2=1$ so we have

$\vec{v}\cdot\vec{n} = x + x - 2x = 0$ as advertised.

Note POLAR COORDS.: $\vec{\nabla}u = \underset{\text{unit radial}}{\frac{\partial u}{\partial r}\vec{e_r}} + \underset{\text{unit angular}}{\frac{1}{r}\frac{\partial u}{\partial \theta}\vec{e_\theta}}$

and on circle, $\vec{n} = \vec{e_r}$. Also $\vec{e_r}\cdot\vec{e_\theta} = 0$ everywhere.

So we need $\left(\frac{\partial u}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\vec{e_\theta}\right)\cdot\vec{e_r} = \frac{\partial u}{\partial r} = \cos\theta - \frac{1}{r^2}\cos\theta$

For $r^2=1$, we get 0.

$\boxed{3.3.19}$ $\frac{\delta P}{\delta u} = \iint_S v\,[?]\,dx\,dy + \int_C v\,[??]\,ds$

from $\iint\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - fv\right]dx\,dy$ with Green's formula $\left(\begin{matrix}u \to v\\ w \to \nabla u\end{matrix}\right.$

yields $\iint_S v\left[-\underbrace{\operatorname{div}\operatorname{grad} u}_{\displaystyle -\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}}\right]dx\,dy + \int_C v(\vec{\nabla}u\cdot\vec{n})\,ds$

Diff. eq$^{\underline{n}}$ $\quad -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - f = 0$ in S

B.C. $\quad \vec{\nabla}u\cdot\vec{n} = 0$ on C