MATH 413 – HOMEWORK 7 – SOLUTION

3.2.1 $\frac{d}{dt}P(u+tv)\Big|_{t=0}$ gives linear term (in v) in $P(u+v)$

$$= \frac{d}{dt}\frac{1}{2}\int_0^1 \left(\frac{d^2(u+tv)}{dx^2}\right)^2\Bigg|_{t=0} dx = \frac{1}{2}\int_0^1 2\,\frac{d^2u}{dx^2}\,\frac{d^2v}{dx^2}\,dx$$

WEAK FORM

$$= \int_0^1 \frac{d^2u}{dx^2}\,\frac{d^2v}{dx^2}\,dx = \int_0^1 \frac{d^4u}{dx^4}\,v\,dx + \Big[u''v' - u'''v\Big]_{x=0}^{x=1}$$

strong form

integration by parts

so $\frac{\delta P}{\delta u} = \frac{d^4u}{dx^4}$ with appropriate boundary conditions.

3.2.2 $u(0)=0,\ u(1)=0 \rightarrow P(u) = \int_0^1\left[\frac{1}{2}\left(\frac{du}{dx}\right)^2 + x\,u(x)\right]dx$

minimized when $\frac{\delta P}{\delta u} = 0$ so $\frac{d}{dt}P(u+tv)\Big|_{t=0} = \int_0^1 \frac{du}{dx}\frac{dv}{dx} + x\,v(x)\,dx$
$= 0$

for all v with $v(0)=0,\ v(1)=0$ – integrate by parts, get

$$\int_0^1\left[-\frac{d^2u}{dx^2} + x\right]v(x)\,dx = 0 \quad \text{so} \quad -\frac{d^2u}{dx^2} + x = 0.$$

Thus $u'' = +x \Rightarrow u = +x^3/6 + Ax + B$; $u(0)=0 \Rightarrow B=0$
$u(1)=0 \Rightarrow A = -\frac{1}{6}$

Solution: $u(x) = \frac{1}{6}(x^3 - x^3)$

3.2.10 $u'''' = 1 \qquad u = x^4/24 + A + Bx + Cx^2 + Dx^3$

$u(0)=0,\ u'(0)=0 \Rightarrow A, B = 0$; $u(1) \overset{=0}{,}\ u'(1) = 0$ gives

$\frac{1}{24} + C + D = 0\ ,\ \frac{1}{6} + 2C + 3D = 0 \Rightarrow D = -\frac{1}{12},\ C = \frac{1}{24}$

So $u(x) = \frac{1}{24}x^2 - \frac{1}{12}x^3 + \frac{1}{24}x^4 = \boxed{\frac{x^2(1-x)^2}{24}}$

$\boxed{3.2.13}$ Slope $= n$ $\quad - \quad \int_0^1 S_n' \, dx = n \cdot \frac{1}{n} = 1$

But $\int_0^1 (S_n')^2 dx = n^2 \cdot \frac{1}{n} = n \to +\infty$

so $\int \delta \, dx = 1$ but $\int \delta^2 dx = +\infty$ (formally)

$\boxed{3.6.1}$ $P(u) = \int \left[\frac{1}{2} c(x)(u''(x))^2 - f(x)\, u(x)\right] dx$

$\frac{d}{dt} P(u+tv)\Big|_{t=0} = \int \left[c(x) \cdot u''(x)\, v''(x) - f(x)\, v(x)\right] dx = 0$

for all "admissible" v

Weak form

Strong form Integrate by parts (with boundary conditions)

to get $\left[c(x)\, u''(x)\right]'' - f(x) = 0$

$\boxed{3.6.3}$ (a) $\int (u')^2 + e^u \, dx$ $\qquad \frac{d}{dt} P(u+tv)\Big|_{t=0} = 0$ for admissible v

yields $\int \left[2u'v' + e^u v\right] dx = 0$

Integrating by parts: $\int \left[-2u'' + e^u\right] v \, dx = 0$

so $-2u'' + e^u = 0$

(b) $\int \frac{d}{dt}(u+tv)(u'+tv')\,dx = \int (uv' + u'v)\,dx$

$= uv\Big|_a^b \to 0$ with typical B.C.

Note $P(u) = \int uu'\,dx = \int (\tfrac{1}{2}u^2)'\,dx = \tfrac{1}{2}[u^2(b) - u^2(a)]$ for any u – no integral needed

(c)

$P(u) = \int x^2 (u')^2\,dx$ yields $\frac{d}{dt} P(u+tv)\Big|_{t=0} = \int 2x^2 u' v'\,dx$

so int. by parts implies: $-\frac{d}{dx}(2x^2 u') = 0$

$\boxed{3.6.15}$ (a) $\iint \left(\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2}\right) dx\,dy$

Do $\frac{d}{dt} P(u+tv)\Big|_{t=0} = 0$ for all v admissible $\to$ integrate by parts in x, y or both

Get $\iint \left(\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial^2 x \partial^2 y} + \frac{\partial^4 u}{\partial y^4}\right) v\,dx\,dy = 0$

Strong form of Euler: $\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial^2 x \partial^2 y} + \frac{\partial^4 u}{\partial y^4} = 0$

[Biharmonic eqn]

(b) $\iint [y\, u_x v_x + u_y v_y]\,dx\,dy = 0 \Rightarrow$

$-\frac{\partial}{\partial x}(y u_x) - \frac{\partial}{\partial y}(u_y) = 0$

(c) $\int v\sqrt{1+(u')^2} + u \cdot \tfrac{1}{2}(1+(u')^2)^{-1/2} \cdot 2u'v' \,dx = 0$

or $\int v \left\{ \left[-uu'(1+(u')^2)^{-1/2}\right]' + \sqrt{1+(u')^2} \right\} dx = 0$

so $$-\left(\frac{uu'}{\sqrt{1+(u')^2}}\right)' + \sqrt{1+(u')^2} = 0$$

(d) $P(u)$ with $\iint u^2\,dx\,dy = 1 \rightarrow \dfrac{\delta P}{\delta u} = \lambda \dfrac{\delta Q}{\delta u}$.

$\dfrac{\delta P}{\delta u} = -u_{xx} - u_{yy}$ so with $\frac{1}{2}$ incorporated into $Q(u) = \frac{1}{2}\left[\iint u^2\,dx\,dy - 1\right] = 0$

gives $-u_{xx} - u_{yy} = \lambda u$ as in answer key

[I might have $2u = \dfrac{\delta Q}{\delta u}$ without $\frac{1}{2}$ in Q]