MATH 413
HOMEWORK 4 SOLN

2.1.2

(a) $A_0 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ so $A_0^T A_0 = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$

$$A_0^T C A_0 = \begin{bmatrix} c_1+c_2+c_5 & -c_1 & -c_2 & -c_5 \\ -c_1 & c_1+c_3+c_4 & -c_3 & -c_4 \\ -c_2 & -c_3 & c_2+c_3+c_6 & -c_6 \\ -c_5 & -c_4 & -c_6 & c_4+c_5+c_6 \end{bmatrix}$$

(b) A^TCA is as in text; $A^TA = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ $(C=I)$

(c) $A^TA = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -1 \\ 0 & 8/3 & -4/3 \\ 0 & -4/3 & 8/3 \end{bmatrix}$

$\sim \begin{bmatrix} 3 & -1 & -1 \\ 0 & 8/3 & -4/3 \\ 0 & 0 & 2 \end{bmatrix}$ all positive pivots.

<u>or</u> $3\begin{pmatrix} 1 \\ -1/3 \\ -1/3 \end{pmatrix}\begin{pmatrix} 1 & -1/3 & -1/3 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 1/3 & 1/3 \\ -1 & 1/3 & 1/3 \end{pmatrix}$

so

$$A^TA = 3\ell_1\ell_1^T + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 8/3 & -4/3 \\ 0 & -4/3 & 8/3 \end{pmatrix} = 3\,\ell_1\ell_1^T + \tfrac{8}{3}\begin{pmatrix} 0 \\ 1 \\ -\tfrac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 & -\tfrac{1}{2} \end{pmatrix}$$

$$+ 2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

$\boxed{2.14}$

$$\begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

becomes

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{so} \quad A^Ty = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and $y + Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ becomes

$$-y_1 - y_2 - y_5 = 0$$
$$y_1 - y_3 - y_4 = 0$$
$$y_2 + y_3 - y_6 = 0$$

Solve for y_4, y_5, y_6 in terms of others:

$$y_5 = -y_1 - y_2$$
$$y_4 = y_1 - y_3$$
$$y_6 = y_2 + y_3$$

6 eq=s in $y_1, y_2, y_3, x_1, x_2, x_3$:

$$y_1 + x_1 + x_2 = 0$$
$$y_2 - x_1 + x_3 = 0$$
$$y_3 - x_2 + x_3 = 0$$
$$y_4 - x_2 = 1$$
$$y_5 - x_1 = 1$$
$$y_6 - x_3 = 1$$

Eliminate y's e.g. using y_4, y_5, y_6 eq=s

Get $\cancel{2x_1 + x_2 + x_3 \quad x_2 = 2x_1 + x_3 = 1}$

$+[y_1 + y_2 + y_5] = 0 \Rightarrow +3x_1 + x_2 + x_3 = -1$

And similarly $y_1 - y_3 - y_4 = 0 \Rightarrow$

$$-x_1 + 3x_2 - x_3 = -1$$

and

finally $-x_1 - x_2 + 3x_3 = -1$

which we know is $+A^TCA\, x = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$ which has unique sol^{n}.

Also eq^{n}s are symmetric in $x_1 \to x_2 \to x_3$ permutations so $x_1 = x_2 = x_3 = -1$

whereby $y_4 = y_5 = y_6 = 0$ and $y_1 = y_2 = y_3 = 0$ also.

$\boxed{2.1.9}$ (i) By "rules" (5), (6) we find

$$A^TCA = \begin{pmatrix} c_1+c_2 & -c_1 & -c_2 \\ -c_1 & c_1+c_3+c_4 & -c_3 \\ -c_2 & -c_3 & c_2+c_3+c_5 \end{pmatrix}$$

(ii) column-row $\quad C = \text{diag}(c_1, c_2, c_3, c_4, c_5)$

$$A^TCA = c_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}(-1\ 1\ 0) + c_2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}(-1\ 0\ 1)$$

$$+ c_3\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}(0\ -1\ 1) + c_4\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}(0\ -1\ 0) + c_5\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}(0\ 0\ -1)$$

$$= \begin{pmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} c_2 & 0 & -c_2 \\ 0 & 0 & 0 \\ -c_2 & 0 & c_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{pmatrix}$$

$$+\begin{pmatrix}0&0&0\\0&c_4&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0\\0&0&0\\0&0&c_5\end{pmatrix} \quad ✓$$

$\boxed{2.2.3}$ (a) $Q = \frac{1}{2}(y_1^2+y_2^2)$ w/ $y_1 - y_2 = 1$

$\nabla Q = \lambda \nabla(y_1 - y_2 - 1) \Leftrightarrow y_1 = \lambda$, $y_2 = -\lambda$ $\quad y_1 - y_2 = 1 \Rightarrow 2\lambda = 1$, $\lambda = \frac{1}{2}$

$(\frac{1}{2}, -\frac{1}{2}) \rightarrow$ value of Q is $\frac{1}{2}\left[\frac{1}{4}+\frac{1}{4}\right] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

(b) $\nabla Q = x_1 \nabla(y_1 - y_2 - 1) + x_2 \nabla(y_2 - y_3 - 2)$

$y_1 = x_1$
$y_2 = -x_1 + x_2$ $\quad$ $> y_1 - y_2 = 1$, $> y_2 - y_3 = 2$ $\quad$ implies $2x_1 - x_2 = 1$, $-x_1 + 2x_2 = 2$
$y_3 = -x_2$

so $3x_2 = 5$, $x_2 = 5/3$
$x_1 = 4/3$

$y_1 = 4/3$
$y_2 = 1/3$ $\quad$ constraints - ok. $\quad Q = \frac{1}{2}\left[\frac{16+1+25}{9}\right] = \frac{21}{9} = 7/3$
$y_3 = -5/3$

(c) $Q = y_1^2 + y_1 y_2 + y_2^2 + y_2 y_3 + y_3^2 - y_3$ with $y_1 + y_2 = 2$

$\nabla Q = \lambda \nabla[y_1 + y_2 - 2]$

$2y_1 + y_2 = \lambda$ with $y_1 + y_2 = 2$ so $\quad 4 - y_2 = \lambda$

$y_1 + 2y_2 + y_3 = \lambda$ $\quad 2 + y_2 + y_3 = \lambda$

$y_2 + 2y_3 - 1 = 0$ $\quad y_2 + 2y_3 = 1$

$y_2 = 4 - \lambda$
$y_3 = -6 + 2\lambda$
$-8 + 3\lambda = 1$, $\lambda = 9/3 = 3$

so $y_2 = 4-3 = 1$
$y_3 = -6+6 = 0$
$y_1 = 1$

$Q = 1^2 + 1\cdot 1 + 1^2 + 0 = 3$

(d) $\nabla Q \Rightarrow \begin{pmatrix} y_1+y_2 \\ y_1-1 \end{pmatrix} = \lambda \nabla(y_1+y_2) = \begin{pmatrix} \lambda \\ \lambda \end{pmatrix}$

$y_1 + y_2 = \lambda$, $y_1 - 1 = \lambda$ $\to$ $\lambda = 0,\ y_1 = 1,\ y_2 = -1$ Q there is $\frac{1}{2}(1-2)+1$

But Indefinite: if $y_2 = -y_1$, Q becomes a fn of y_2 only

$Q = \frac{1}{2}\left[(-y_2)^2 + 2(-y_2)y_2\right] - y_2 = \frac{1}{2}\left[y_2^2 - 2y_2^2\right] - y_2$
$\to -\infty$ as $y_2 \to \pm\infty$

$\boxed{2.2.7}$ $A^T y = y_1 + 2y_2 + 2y_3 = 18$

$A = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 18 \end{pmatrix}$

from $-A^TAx = 18$ we get $x = -2$ so $y_1 = 2, y_2 = 4, y_3 = 4$
distance is $\sqrt{4+16+16} = \sqrt{36} = 6$

$\boxed{2.2.8}$ (i) $(2,4,4)$ – length $\sqrt{36} = 6$

(ii) $A^Ty \le \|A\|\|y\|$ so $18 \le 3\|y\|$ $\|y\| = 6$