SOLUTION

Math 316 – Exam 2 – Spring 2008 – Professor Sachs

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Answer each question on this paper. Read them carefully and answer clearly. Exam ends at 5:45pm. The GMU Honor Code is in effect. The exam is worth 100 points.

1. (10 points) For the function $f(x, y, z) = x^2 + 2yz$, find the total derivative $Df\mathbf{a}$ of f at the point $\mathbf{a} = (1, 2, 3)$.

Either $df = f_x\,dx + f_y\,dy + f_z\,dz = 2x\,dx + 2z\,dy + 2y\,dz$

which is $2\,dx + 6\,dy + 4\,dz$ or equivalent

2. (15 points) Define **differentiability** of a function f and also **directional derivatives.** Explain the difference between the two definitions by considering $f(x, y) = xy^2/(x^2 + y^2)$ unless $x = y = 0$, with $f(0, 0) = 0$, which has direction derivatives $D_{\mathbf{v}}f(0, 0) = f(\mathbf{v})$, but is not differentiable (explain why).

f is differentiable at a point (a,b) if $\exists$ a linear mapping (Df) so that

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - f(a,b) - Df_{(a,b)}\langle x-a, y-b\rangle}{\|(x,y) - (a,b)\|} = 0$$

Directional derivatives may exist without the function being differentiable — BUT NOT VICE-VERSA — since differentiability requires the derivatives to match up as a linear map.

Example has typo → best to recall example from class: $r \sin 2\theta = \begin{cases} \dfrac{2xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

3. (10 points) Let f, $g : \mathbf{R} \mapsto \mathbf{R}$ be twice differentiable. Prove that $u(x,t) := f(x-t) + g(x+t)$ satisfies the wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

typos

let $\xi = x-t$, $\eta = x+t$ then $u(x,t) = f(\xi) + g(\eta)$, $\frac{\partial \xi}{\partial x} = 1$, $\frac{\partial \xi}{\partial t} = -1$, $\frac{\partial \eta}{\partial x} = 1$

so $u_{tt} = f''(\xi) + g''(\eta)$ since $u_t = f'(\xi)\cdot(-1) + g'(\eta)\cdot(+1)$ $\frac{\partial \eta}{\partial t} = 1$

$u_{xx} = f''(\xi) + g''(\eta)$ since $u_x = f'(\xi)\cdot 1 + g'(\eta)\cdot 1$

4. (15 points) Describe for a **general** situation what the chain rule says and why it is a sensible result. The general situation is $F : R^n \to R^m$, $G : R^m \to R^k$, both of which are differentiable.

at $\vec{a}$ with $F(\vec{a}) = \vec{b}$ we have that the composite $G \circ F$ is differentiable and $D[G(F)](\vec{a}) = [DG(\vec{b})][DF(a)]$ ← matrix multiplication or composition of linear mappings.

It makes sense because $G(F(\vec{a}+\vec{h})) - G(F(\vec{a}))$

$\approx DG(\vec{b})[F(\vec{a}+h) - F(\vec{a})]$

$\approx DG(\vec{b})\, DF(\vec{a})\, h$ using linear approximation.

5. (10 points) Find the minimum of $x+y+z$ subject to the constraint $1/x+2/y+1/z=1$ where x, y, z are all positive.

Let $f = x+y+z$

$\vec{\nabla} f = \langle 1,1,1 \rangle$; if $g = \frac{1}{x}+\frac{2}{y}+\frac{1}{z}-1$ then $\vec{\nabla} g = \langle -\frac{1}{x^2}, -\frac{2}{y^2}, -\frac{1}{z^2} \rangle$

So $\vec{\nabla} f = \lambda \vec{\nabla} g \Leftrightarrow 1 = -\frac{\lambda}{x^2}, \; 1 = -\frac{2\lambda}{y^2}, \; 1 = -\frac{\lambda}{z^2}$ ← These imply $x^2 = z^2 = y^2/2$

So $x = z = y/\sqrt{2}$ — constraint: $\frac{1}{x} + \frac{\sqrt{2}}{x} + \frac{1}{x} = 1$ so $x = 2+\sqrt{2}$

$y = 2\sqrt{2}+2$

$z = 2+\sqrt{2}$

$$\boxed{x+y+z = 4\sqrt{2}+6}$$

6. (15 points) Consider a function $f(x,y)$. Describe conditions that should hold at a local **maximum** and explain why what you claim is correct. Then show that the function $f(x,y) = 4xy - x^4 - y^4$ satisfies these conditions at $(-1,-1)$.

At local max, $\vec{\nabla} f = \vec{0}$ or DNE ; if $f \in C^2$, Hessian matrix $D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$ must be negative definite

$f = 4xy - x^4 - y^4 \Rightarrow \vec{\nabla} f = \langle 4y - 4x^3, 4x - 4y^3 \rangle$

$\vec{0}$ at $(-1,-1)$

$H = D^2 f = \begin{pmatrix} -12x^2 & 4 \\ 4 & -12y^2 \end{pmatrix} = \begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix}$ at $(-1,-1)$

$\det = \lambda_1 \lambda_2 = 12^2 - 4^2 > 0$, $-24 = \operatorname{tr} H = \lambda_1 + \lambda_2$ so λ_1, λ_2 both negative.

OR Sub determinants

7. (10 points) Give a careful statement of the Inverse Function Theorem. Find conditions on (x_0, y_0) so that the mapping

$$s = x^2 + y^2, \ t = 1/(xy)$$

has an inverse near (x_0, y_0).

(s,t) can be solved for (x,y) if $\frac{\partial(s,t)}{\partial(x,y)}$ is invertible.

$$\frac{\partial(s,t)}{\partial(x,y)} = \begin{pmatrix} 2x & 2y \\ -\frac{1}{x^2y} & -\frac{1}{y^2x} \end{pmatrix}$$ so invertible iff $2x\left(-\frac{1}{y^2x}\right) - \left(-\frac{1}{x^2y}\right)(2y) \neq 0$

$-\frac{1}{y^2} + \frac{1}{x^2} \neq 0$ so $x \neq, y \neq 0$ and $x^2 \neq y^2$

INV FN. THM: If $D\vec{F}\big|_{(x_0,y_0)}$ is invertible for a C^1 function in nbhd of (x_0,y_0) then $\exists$ inverse fn. $\vec{G}$ which is C^1 and $D\vec{G}\big|_{F(x_0,y_0)} = [DF|_{(x_0,y_0)}]^{-1}$ works!

(see text)

8. (15 points) Show that the relation $G(x, y) = y^3 + y - x^2 = 0$ defines y as an implicit function of x near $(\sqrt{2}, 1)$. What is the derivative of y as a function of x at $(\sqrt{2}, 1)$?

$G_x = -2x, \quad G_y = 3y^2 + 1 \longrightarrow$ at $(\sqrt{2}, 1)$, $G_y \neq 0$

so can find $\frac{dy}{dx}$ and know y is locally a fn of x.

From

$dG = G_x\, dx + G_y\, dy = 0$ we infer $\frac{dy}{dx} = -\frac{G_x}{G_y} = \frac{-2\sqrt{2}}{4}$