SOLUTION

Math 316 – Exam 1 – Spring 2008 – Professor Sachs

Name: REED D. BOOK

Answer each question on this paper. Read them carefully and answer clearly. Exam ends at 5:45pm. The GMU Honor Code is in effect. The exam is worth 100 points.

1. (10 points) Define **pointwise convergence** of a sequence of functions f_n on a set A and also **uniform convergence**. Explain the difference between the two definitions.

See book – For uniform convergence, same N works for all $x \in A$

2. (15 points) Prove that the limit of a uniformly convergent sequence of continuous functions is continuous.

See book – usual $\varepsilon/3$ proof : $f_n \to f$

$$|f(x) - f(a)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|$$

1st, 3rd terms $< \varepsilon/3$ for $n \geq N$ (fixed) by unif. conv.

2nd term : $\exists\, \delta$ (by continuity) s.t. $|f_n(x) - f_n(a)| < \varepsilon/3$

3. (10 points) Consider the sequence of functions $f_n(x) = \frac{1}{n^4x^4+1}$. Find the pointwise limit for each x and show the convergence is not uniform on the interval $[-1, 1]$.

If $x \neq 0$, $f_n(x) = \frac{1}{n^4x^4+1} = \frac{1}{n^4}\left(\frac{1}{x^4 + 1/n^4}\right) \to 0$ as $n \to \infty$

If $x = 0$, $f_n(0) = \frac{1}{0+1} = 1$ for all n so $f_n(0) \to 1$

Pt w'se limit is $\begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$

EITHER: violates problem 2 or use $\varepsilon = \frac{1}{2}$ (or something similar)

Find $x_n \to 0$ s.t. $f_n(x_n) = \frac{1}{2}$ — non-uniform since for every n, $|f_n(x_n) - f(0)| \geq \frac{1}{2}$ and $\{x_n\}$ intersects all δ nbhds of 0

4. (15 points) Describe the Weierstrass M-test for proving uniform convergence of a sum of functions $\sum_{k=1}^{\infty} f_k(x)$. Then apply it to show that the series $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^4}$ is uniformly convergent for $0 \leq x \leq 2\pi$

Usual: If $|f_k(x)| \leq M_k$ and $\sum_k M_k < \infty$ then $\sum f_k(x)$ converges uniformly (and absolutely)

$|\sin(kx)| \leq 1$ so $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^4}$ converges uniformly using $M_k = \frac{1}{k^4}$

SOME OF YOU FORGOT $|\ |$ PART

5. (10 points) Find the Taylor series for the functions $\sin x$ and $\cos x$ in powers of $x - \pi$ and verify that the derivative series for sin is cos.

Sin x has value 0 at π, deriv. : $\cos \pi = -1$ etc.

Series $-(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \dots$

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cos x : $-1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \dots$

Derivative of sin series is cos series.

6. (15 points) Define the limit of a sequence in a metric space. Then show that (a) a sequence can have at most one limit and (b) every convergent sequence is a Cauchy sequence.

Limit of sequence x_n is a if given any $\varepsilon > 0$, $\exists N$ such that $\rho(x_n, a) < \varepsilon$ for all $n \geq N$.

↑

distance in metric

Now (a) At most one limit (Hmwk) : Either by contradiction (assume two — $a \neq b$) — then for $\varepsilon = \rho(a,b)/3$ ← or similar, we require $\rho(x_n, a) < \rho(a,b)/3$, $\rho(x_n, b) < \rho(a,b)/3$ which contradicts triangle inequality

or direct $\rho(a,b) \leq \rho(x_n, a) + \rho(x_n, b) < 2\varepsilon$ for all $\varepsilon > 0$ So $\rho(a,b) = 0$.

(b) To show : given $\varepsilon > 0$, $\exists N$ s.t. $\rho(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. Use limit def. with $\varepsilon/2$ and triangle inequality: $\rho(x_n, x_m) \leq \rho(x_n, a) + \rho(x_m, a) < \varepsilon$ for all $n, m \geq N$.

7. (10 points) Describe the difference between a norm and a metric – is one a special case of the other? What features are particular to the more specialized setting?

Norm: Linear spaces, scales so $\rho(u,v) = \|u-v\|$ in that setting and then $\|\alpha u\| = |\alpha|\,\|u\|$: special feature of norm.

some forgot this

Metric More general notion — for instance the strange discrete metric has no scaling property.

8. (15 points) Show that the space of continuous functions on a closed bounded interval $[a,b]$ forms a normed linear space under the proposed norm:

$$||f|| := sup_{a\leq x\leq b}|f(x)|.$$

Then relate convergence in this norm for a sequence of functions to one of our notions of convergence for sequences of function, explaining your reasoning.

If f, g are in $C[a,b]$ then $\|f+g\| = \sup_x |f(x)+g(x)|$

$\leq \sup_x |f(x)| + \sup_x |g(x)|$

$= \|f\| + \|g\|$ (triangle inequality)

Also norm is finite for all $f \in C[a,b]$ by extreme value theorem.

Clearly $\|f\| \geq 0$ — also $\|f\| = 0 \Rightarrow f(x) \equiv 0$ $(\sup|f| = 0)$

Finally: This is uniform convergence restricted to continuous functions, since a sequence $f_n \to f$ in this norm means $\sup_x |f_n(x) - f(x)| \to 0$

Or else: given $\varepsilon > 0$, $\exists N$ s.t. $\sup_x |f_n(x) - f(x)| < \varepsilon$ for all $n > N$ which is uniform convergence since N works for all x. (see #1)