MATH 316 — HOMEWORK #2 SOLUTION

$\boxed{8.2}$ #4 $\det \begin{pmatrix} x-a_1 & y-a_2 & z-a_3 \\ b_1-a_1 & b_2-a_2 & b_3-a_3 \\ c_1-a_1 & c_2-a_2 & c_3-a_3 \end{pmatrix} = 0$

We know the determinant is 0 when any row/column is 0 and also when rows/columns are equal. So this function is linear in x, y, z and 0 at $(x,y,z) = (a_1, a_2, a_3)$ (0 row) and $(x,y,z) = (b_1, b_2, b_3)$ and $(x,y,z) = (c_1, c_2, c_3)$ by equal rows. Since the points are not collinear, the linear function is non-degenerate (usual $(\vec{b}-\vec{a}) \times (\vec{c}-\vec{a}) \neq \vec{0}$ shows rank $= 2$)

#11 $M := \sup_{\|x\|=1} \|T(x)\|$ (a) Show $M \leq \|T\| = \inf\{C > 0 : \|T(x)\| \leq C\|x\|\}$

~~By Thm 8.17~~

$\|T(x)\| \leq \|T\| \, \|x\|$ so for all $\|x\|=1$, $\|T(x)\| \leq \|T\|$

therefore $M = \sup_{\|x\|=1} \|T(x)\| \leq \|T\|$

(b) $\|T(x)\|$ for $\vec{x} \neq \vec{0}$ — write $\vec{x} = \|\vec{x}\| \vec{u}$ where $\|\vec{u}\| = 1$

Then $\|T(x)\| = \|T(\|x\|\vec{u})\| = \|x\| \|T(u)\|$

$\leq \|x\| \cdot M$ so $M \geq \dfrac{\|T(x)\|}{\|x\|}$ for all $\vec{x} \neq \vec{0}$

Thus $\|T(x)\| \leq M\|x\|$

(c) By definition of $\|T\|$ as inf, $\|T\| \leq M$. With (a), that yields $\|T\| = M$.

(d) As noted in (b), $\dfrac{\|T(x)\|}{\|x\|}$ is indep of scale so $\sup_{\vec{x} \neq \vec{0}} \dfrac{\|T(x)\|}{\|x\|}$

$= \sup_{\|\vec{x}\|=1} \|T(x)\|$

10.1 (#3) Prove theorem 10.14

(i) If $x_n \to L_1$ and $x_n \to L_2$ then for $\varepsilon = \frac{\rho(L_1, L_2)}{3} > 0$
we must have $\rho(x_n, L_1) < \varepsilon$, $\rho(x_n, L_2) < \varepsilon$ for all $n \geq N$
which violates the triangle inequality

(ii) For any $\varepsilon > 0$, $\exists N$ s.t. $\rho(x_n, a) < \varepsilon$ for all $n \geq N$.
Subsequences x_{n_k} can use $k \geq N$ also [less points included]
same Limit

(iii) let $\varepsilon = 1$ Then for some N, $\rho(x_n, a) < 1$ for all $n \geq N$.
let $M = \max\{\rho(x_1, a), \rho(x_2, a), \dots, \rho(x_N, a), \text{limit} + 1\}$ - done.

(iv) Given any $\varepsilon > 0$, $\exists N$ s.t. $\rho(x_n, a) < \varepsilon/2$ for all $n \geq N$.

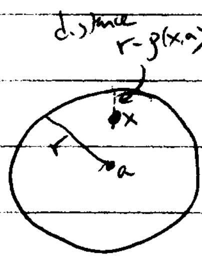
Then $\rho(x_n, x_m) < \varepsilon$ by triangle inequality.

(#9) (a) $x \in B_r(a)$. let $\varepsilon = \frac{r - \rho(x, a)}{2}$

Then for all y with $\rho(x, y) \leq \varepsilon$, it follows
that $\rho(y, a) \leq \rho(x, y) + \rho(x, a) \leq \frac{r - \rho(x, a)}{2} + \rho(x, a)$

$\leq \frac{r + \rho(x, a)}{2} < r$ since $\rho(x, a) < r$

Picture



(b) $a \neq b$ - use $r = \frac{\rho(a, b)}{3}$ (gap)
can't have $\rho(x, a) < r$ and $\rho(x, b) < r$ (triangle inequality)

(c) If $x \in B_r(a) \cap B_s(b)$ then 

then $\rho(x,a) < r$, $\rho(x,b) < s$. As in (a) use half of distance

Let $c = \min\left(\frac{r-\rho(x,a)}{2}, \frac{s-\rho(x,b)}{2}\right)$ then $B_c(x) \subseteq B_r(a) \cap B_s(b)$

Also by triangle inequality, for any $x_1 \in B_r(a)$, $\rho(x,x_1) < 2r$

For $B_s(b)$, $\rho(x, x_2) < 2s$ so with $d = \max(2r, 2s)$, $B_d(x) \supseteq B_r(a) \cup B_s(b)$

$\boxed{10.2}$ #5 Prove Th$^{\underline{m}}$ 10.26

(i) If $f(x) = g(x)$ for all $x \in X \setminus \{a\}$ then

must show $0 < \rho(x,a) < \delta$ implies $\tau(g(x), L) < \varepsilon$

But this holds for f and at all pts like these, $f = g$ so use same δ.

(ii) First if $x_n \in X \setminus \{a\}$ and $x_n \to a$ then by limit of f, ~~given~~ any ε, $\exists$ $\delta > 0$ s.t. $0 < \rho(x,a) < \delta$ implies $\tau(f(x), L) < \varepsilon$.

But since $x_n \to a$, $\exists$ N s.t. $\rho(x_n, a) < \delta$ for all $n \geq N$. Therefore for all such $n \geq N$, $\tau(f(x_n), L) < \varepsilon$.

Conversely, suppose $f(x_n) \to L$ for all sequences converging to a, but ~~limit~~ does not exist. Then for some ε_0, no δ works. Using sequence $\delta_n \to 0$ (like $1/n$ or $\frac{1}{2^n}$) we have points $x_n \in X \setminus \{a\}$ with $\rho(x_n, a) < \delta_n$ and $\tau(f(x_n), L) \geq \varepsilon_0$. But $x_n \to a$ so this is a contradiction.

(iii) In $\mathbb{R}^n$, addition and metric imply (with $L = \lim_{x \to a} f(x)$, $M = \lim_{x \to a} g(x)$)

$\lim_{x \to a} (f+g)(x) = L + M$

since $|f(x) + g(x) - L - M| \leq |f(x) - L| + |g(x) - M| < \varepsilon$ (usual $\varepsilon/2$)

$\lim_{x \to a} (\alpha f)(x)$ if $\alpha = 0$, trivial; for $\alpha \neq 0$ use $|\alpha f(x)| = |\alpha|\,|f(x)|$

so $|\alpha f(x) - \alpha M| = |\alpha|\,|\cancel{f_n}\, - M| < \varepsilon$ if $|f(x) - M| < \varepsilon/|\alpha|$

new $\tilde{\varepsilon}$

Similarly $\underset{\text{~~dot product~~}}{f(x) \cdot g(x)} - M \cdot N = (f(x) - M) \cdot N$

$\qquad + (N - g(x)) \cdot f(x)$

use triangle inequality and Cauchy-Schwarz (assume $\vec{N} \neq \vec{0}$; if $\vec{N} = \vec{0}$, easier)

$$|f(x) \cdot g(x) - M \cdot N| \leq |f(x) - M|\,|N| + |f|\,|N - g| < \varepsilon$$

if $|N| > 0$, use $|f(x) - M| < \min\left(\frac{\varepsilon}{2|N|}, 1\right)$, $|g(x) - N| < \frac{\varepsilon}{2(|M|+1)}$

usual fudge ~~for products~~

On $\mathbb{R}$ with $g(x) \to M \neq 0$: Can guarantee $g(x) \neq 0$ for $\rho(x, c) < \delta_1$;

using $|M|/2 = \varepsilon$,

Then $\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| = \left|\frac{Mf(x) - Lg(x)}{g(x)\,M}\right|$ $\qquad |1/g(x)| \leq \frac{2}{|M|}$

$$\leq \frac{|Mf(x) - Lg(x)|\; 2}{|M|^2}$$ reduces to addition

(iv) Squeeze $Y = \mathbb{R}$ $\quad g(x) \leq h(x) \leq f(x)$ - pick

δ_1 so $|g(x) - L| < \varepsilon$, δ_2 so $|f(x) - L| < \varepsilon$ then

$\delta = \min(\delta_1, \delta_2)$ guarantees $|h(x) - L| < \varepsilon$ also for

$\rho(x, a) < \delta$.

(v) COMPARISON $f(x) \le g(x)$ for all $x \in X \setminus \{a\}$ and limits exist then $L \le M$: Suppose not — $L > M$ pick $\varepsilon = \frac{L-M}{2}$ Then for a common δ working for both f, g limits,

$$f(x) > L - \left(\frac{L-M}{2}\right) = \frac{L}{2} + \frac{M}{2} = M + \varepsilon > g(x)$$

for all $x \in X \setminus \{a\}$ with $\rho(x,a) < \delta$ — this is a contradiction

#9 (a) E bounded in X implies convergent subsequence in E. But E closed says limit must be in E.

(b) show f continuous, E closed, bdd then f must be bounded.:

Proof If not, $\exists\, x_n \in E$ s.t. $|f(x_n)| > n$ for all n.

For a convergent subsequence $x_{n_k} \to a$ in E, $|f(a)|$ must be $+\infty$ — impossible for f continuous.

(c) Either use book's proof that sup / inf are achieved OR what we did before — sequences with f values tending to sup / inf. Then subsequences converge, limit takes on desired extreme values.