HOMEWORK #1 SOLUTION

MATH 316 SPRING 2008 Prof SACHS

① $f_n(x) = \dfrac{1}{n^2x^2+1}$ with $f_n'(x) = -\dfrac{2n^2x}{(n^2x^2+1)^2}$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n^2}\left(\dfrac{1}{x^2+1/n^2}\right) = 0$ unless $x=0$; $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 1 = 1$

$f_n \to f$ but not uniformly: (a) Since limit isn't continuous but each $f_n(x)$ is

(b) Using $x_n = \frac{1}{n}$: $f_n(x_n) = \frac{1}{1+1} = \frac{1}{2}$ for all n

$x_n \to 0$ so we expected $\lim_{n\to\infty} f_n(0) = 1$.

(c) Direct proof [not as clean]

$\lim_{n\to\infty} \dfrac{-2n^2x}{(n^2x^2+1)^2} = 0$, (even at $x=0$) again using $x_n = \frac{1}{n}$, $\lim_{n\to\infty} f_n'(x_n) = \lim_{n\to\infty} \dfrac{-2n^2\cdot\frac{1}{n}}{(n^2\cdot\frac{1}{n^2}+1)^2} = \lim_{n\to\infty} \frac{-2n}{4} = -\infty$

② Weierstrass: $\frac{1}{k^x} \le \frac{1}{k^a}$ for $x \ge a > 1$

so $\sum_{k=1}^{\infty} \frac{1}{k^x} \le \sum_{k=1}^{\infty} \frac{1}{k^a}$ ← converges by integral test

The term-wise derivative is $\frac{d}{dx}\left(k^{-x}\right) = \frac{d}{dx}\left(e^{(\ln k)\cdot(-x)}\right)$

$= -\ln k \cdot e^{(\ln k)\cdot(-x)} = -\dfrac{\ln k}{k^x}$

Use Theorem ~~7.12~~ 7.14 and an estimate on derivative series to conclude $\sum_{k=1}^{\infty} -\dfrac{\ln k}{k^x}$ is derivative of $\zeta(x)$.

<u>Idea</u> for $1 < a \le x$ we let $p = \frac{1+a}{2} > 1$, compare $+\dfrac{\ln k}{k^x}$ to $\dfrac{1}{k^p}$

since by L'Hopital's rule, $\lim_{k\to\infty} \dfrac{\ln k}{k^b} = 0$ for any $b > 0$.

③ 7.2 problem 8

(a) By ratio test, successive terms $k=l,\ l+1$
$$\left|\frac{(-1)}{(l+1)(n+l+1)}\left(\frac{x}{2}\right)^2\right|$$

$$= \frac{x^2}{4}\cdot\frac{1}{(l+1)(n+l+1)} \longrightarrow 0 \text{ as } l\to+\infty.$$ so radius of conv. $=+\infty$

which implies uniform convergence on any finite interval $[a,b]$

(b) Direct calculation (term-by-term derivatives):

$$x^2 B_n''(x) + x B_n'(x) + (x^2 - n^2) B_n(x)$$

$$= x^2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}(n+2k)(n+2k-1)\frac{x^{n+2k-2}}{2^{n+2k}}$$

$$+ x\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\frac{(n+2k)\,x^{n+2k-1}}{2^{n+2k}} + (x^2-n^2)\sum\frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}$$

3 out of four terms keep power of x as $n+2k$ in end; $x^2 B_n(x)$ goes up by 2 powers of x — needs some bookkeeping.

<u>Some algebra</u> $x^2 B_n''(x) + x B_n(x) + (x^2-n^2)B_n(x)$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}\left\{(n+2k)(n+2k-1) + (n+2k) - n^2\right\}$$

simplify here

$$+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}x^2\left(\frac{x}{2}\right)^{n+2k}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}\left[(n+2k)^2 - n^2\right] + \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n+k)!}\frac{x^{n+2k+2}}{2^{n+2k}}$$

(diff of squares)

Now $(n+2k)^2 - n^2 = (2n+2k)(2k)$ (note: 0 for $k=0$)

so $$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{n+2k} \cdot (2n+2k)(2k)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{n+2k} \cdot 2^2 \cdot k \cdot (n+k) = \sum_{k=1}^{\infty} \frac{(-1)^k \, x^{n+2k}}{(k-1)!\,(n+k-1)!\,2^{n+2k}}$$

drop $k=0$

[now shift with $l = k-1$ say, $k = l+1$] $$= \sum_{l=0}^{\infty} (-1)^{l-1} \frac{x^{n+2l+2}}{l!\,(n+l)!\,2^{n+2l}}$$

differs from other sum only by $-$ sign, so adding, get 0.

i.e. $x^2 B_n''(x) + x B_n'(x) + (x^2 - n^2) B_n(x) = 0$

(c) From series, $$x^n B_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2n+2k}}{k!\,(n+k)!\,2^{n+2k}}$$

and derivative in x is

$$\left(x^n B_n(x)\right)' = \sum_{k=0}^{\infty} \frac{(-1)^k (2n+2k)\, x^{2n+2k-1}}{k!\,(n+k)!\,2^{n+2k}}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2n+2k-1}}{k!\,(n+k-1)!\cdot 2^{n+2k-1}} = x^n B_{n-1}(x)$$

by n-1 series above

(4) $\boxed{7.3, \text{ Problem } 7}$ (a) If $\sum_{k=0}^{\infty} a_k = L$, want to show $\lim_{r \to 1^-} \sum_{k=0}^{\infty} a_k r^k = L$ also.

It is <u>not</u> sufficient to take the limit "inside" the sum.

The idea of the proof of Abel's theorem (Thm 7.27) does the trick! This is summation by parts. [with $R=1$]

FINITE SUMS involving r can be dealt with as limits, so we need to control the "tail" as follows:

FIRST By convergence of $\sum_{k=0}^{\infty} a_k$ to L we know that for any $\tilde{\varepsilon}>0$, $\exists N$ s.t. $\left|\sum_{k=0}^{n} a_k - L\right| < \tilde{\varepsilon}$ for all $n \geq N$

or equivalently for all $n, m \geq N$, $\left|\sum_{k=m}^{n} a_k\right| < \tilde{\varepsilon}$

Also the sequence $\{r^k\}$ is decreasing for $0<r<1$ as k increases.

SECOND We need to show $\left|\sum_{k=m}^{n} a_k r^k\right| < \varepsilon$. By summation by parts

$$\sum_{k=m}^{n} a_k r^k = r^n \sum_{k=m}^{n} a_k + \sum_{k=m}^{n-1} \underbrace{(r^k - r^{k+1})}_{\text{geometric series, ratio } r \text{ (telescoping sum)}} \sum_{j=m}^{k} a_j$$

so $$\left|\sum_{k=m}^{n} a_k r^k\right| \leq r^n \left|\sum_{k=m}^{n} a_k\right| + (r^m - r^n)\left|\sum_{j=m}^{k} a_j\right|$$

$$< \tilde{\varepsilon}(r^n + r^m - r^n) = \tilde{\varepsilon} r^m < \tilde{\varepsilon}$$

we can pick $\tilde{\varepsilon} = \varepsilon$. But $\sum_{k=0}^{n} a_k r^k \to \sum_{k=0}^{n} a_k$ as $r \to 1^-$

so $\lim_{r\to 1^-} \sum_{k=0}^{\infty} a_k r^k$ exists and equals L.

(b) $\lim_{r\to 1^-} \sum_{k=0}^{\infty} (-1)^k r^k = \lim_{r\to 1^-} \frac{1}{1+r} = \boxed{\frac{1}{2}}$

(5) #2, 7.4

(a) $\log(1-x)$: deriv of $\log(1-x)$ is $\frac{1}{1-x}\cdot(-1)$

convergent series (geometric) $(-1)[1+x+x^2+\cdots]$ with radius of convergence 1 so

$$\log(1-x) = \boxed{-x - x^2/2 - x^3/3 - \cdots - \frac{x^n}{n} - \cdots}$$ using Thm. 7.32

(b) $\frac{x^2}{1-x^3}$ — several ways : (i) algebra $\frac{1}{1-x^3} = 1+x^3+x^6+\cdots+x^{3n}+\cdots$ for $-1<x<1$

so $$\frac{x^2}{1-x^3} = \boxed{x^2+x^5+x^8+\cdots+x^{3n+2}+\cdots}$$

(ii) calculus $\frac{d}{dx}\log(1-x^3) = \frac{1}{1-x^3}\cdot -3x^2 = -\frac{1}{3}\left[\frac{x^2}{1-x^3}\right]$

use substitution in previous answer, then derivative. (same answer of course)

(c) $e^x(1+x+x^2+\cdots)$ is analytic by product

coefficients
$$\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{j=0}^{\infty}x^j = \sum_{j,k}\sum\frac{x^{k+j}}{k!} = \sum_{j+k=n}\sum_{k=0}^{n}\frac{x^{k+j}}{k!}$$

$$= \boxed{\sum_{n=0}^{\infty}x^n\left[\sum_{k=0}^{n}\frac{1}{k!}\right]}$$

(d) $$\frac{x^3}{(1-x)^2} = x^3\cdot\frac{d}{dx}\left[(1-x)^{-1}\right] = x^3\left[1+2x+3x^2+\cdots\right] = \boxed{\sum_{k=1}^{\infty}k\,x^{k+2}}$$

each analytic so product is too.

(e) $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} = (1+t)^{-1/2}$, $t=-x^2$ so use binomial + subst. + integration

$$\frac{d}{dx}\arcsin x = \sum_{k=0}^{\infty}\binom{-1/2}{k}(-1)^k x^{2k} \Rightarrow \boxed{\arcsin x = \sum\binom{-1/2}{k}\frac{(-1)^k x^{2k+1}}{2k+1}}$$ using $\arcsin 0 = 0$