MATH 116 — SOLUTION TO HOMEWORK #5 — SECTIONS 7.7

Prob 7 $\displaystyle\int_2^{\infty} \frac{dx}{\sqrt{x}}$ $\quad x^{-1/2}$ has antiderivative $2\cdot x^{1/2}$ so we have

$$\lim_{b\to\infty} \int_2^{b} \frac{dx}{x^{1/2}} = \lim_{b\to\infty} 2x^{1/2}\Big|_0^b = +\infty \quad \text{so DIVERGENT.}$$

Prob. 13 $\displaystyle\int_0^{\infty} x e^{-x^2}\,dx = \lim_{b\to\infty} \int_0^{b} x e^{-x^2}\,dx$ $\quad \underline{e^{-u},\ u=x^2 \text{ so } du = 2x\,dx}$

$$= \lim_{b\to\infty} -\tfrac{1}{2}e^{-x^2}\Big|_0^b = \lim_{b\to\infty}\left[-\tfrac{1}{2}e^{-b^2} + \tfrac{1}{2}\right] = \boxed{\tfrac{1}{2}}$$

#19 $\displaystyle\int_2^{\infty} \frac{x}{(x+2)^2}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{x}{(x+2)^2}\,dx$ $\quad$ either partial fractions or substitution

Partial fractions $\quad \dfrac{x}{(x+2)^2} = \dfrac{A}{(x+2)^2} + \dfrac{B}{x+2}$ so $A + B(x+2) = x \rightarrow B = 1,\ A = -2$

$$\lim_{b\to\infty} \int_2^{b} \frac{-2}{(x+2)^2}\,dx + \underbrace{\int_2^{b} \frac{1}{x+2}\,dx}_{\text{divergent}} = +\infty \quad \underline{\text{divergent}}$$

$\underline{\text{Substitution}}$ $\quad u = x+2,\ du = dx,\ x = u-2$ so (note: $x=2 \Leftrightarrow u=4$)

— same idea as above

$$\int_2^{\infty} \frac{x}{(x+2)^2}\,dx = \int_4^{\infty} \frac{u-2}{u^2}\,du = \int_4^{\infty} \left[\frac{1}{u} - \frac{2}{u^2}\right] du$$

NOTE ALSO: $\underline{\text{limit comparison}}$ $\quad \dfrac{x}{(x+2)^2} \approx \dfrac{x}{x^2} = \dfrac{1}{x} \leftarrow$ divergent.

#47 (a) True — comparison idea → integrals always stay smaller so

so $\displaystyle\lim_{b\to+\infty} F(b)$ exists where $F(b) = \int_0^b f(x)\,dx.$

(b) False
Since $f(x) \to 1$, $\int_0^\infty f(x)\,dx$ diverges. [height ≈ 1 base infinite!]

(c) False
$\int_0^1 x^{-p}\,dx$ exists, $\int_0^1 x^{-q}\,dx$ for $q>p$: example

$\int_0^1 x^{-1/2}\,dx$ ok (p exponent not smaller than -1)

But $\int_0^1 x^{-2}\,dx$ divergent

(d) TRUE
$\int_1^\infty x^{-p}\,dx$ exists so $q>p$ implies for x large $x^q > x^p$ or

$x^{-p} = \frac{1}{x^p}$ tops x^{-q} : $x^{-q} < x^{-p}$ - COMPARISON

(e) TRUE
$\int_1^\infty \frac{dx}{x^{3p+2}}$ exists for $p > -1/3$ power $\frac{1}{x^r}$ needs $r>1$

$3p+2>1$, $3p>-1$, $p>-1/3$ yes.

$\boxed{\#69}$ $\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$ and $\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

($a>0$) Method 1 - EULER (2 for 1 DEAL!) $e^{-ax}(\cos bx + i\sin bx) = e^{-(a-ib)x}$

so $\int_0^M e^{-ax}(\cos bx + i\sin bx)\,dx = \int_0^M e^{-(a-ib)x}\,dx$

$$= \frac{e^{-(a-ib)x}}{-(a-ib)}\Big|_0^M = -\frac{e^{-(a-ib)M}}{(a-ib)} + \frac{e^{-(a-ib)\cdot 0}}{-(a-ib)}$$

Thus $\lim_{M\to\infty}\int_0^M e^{-ax}[\cos bx + i\sin bx]\,dx = \frac{1}{a-ib} = \frac{a+ib}{a^2+b^2}$

Real part gives (a), Imag part gives (b).

Method 2 Use Integral Table (or do it by parts)

with $-a$ in place of a [Formulas 86, 87]

$$\int e^{-ax}\cos bx\,dx = e^{-ax}\left[\frac{-a\cos bx + b\sin bx}{a^2+b^2}\right] + C$$

$$\int e^{-ax}\sin bx\,dx = e^{-ax}\left[\frac{-a\sin bx - b\cos bx}{a^2+b^2}\right] + C$$

so
$$\int_0^M e^{-ax}\cos bx\,dx = e^{-aM}\left[\frac{-a\cos(bM)+b\sin(bM)}{a^2+b^2}\right]$$
$$- e^{-a\cdot 0}\cdot\left[\frac{-a\cdot 1 + b\cdot 0}{a^2+b^2}\right]$$

so
$$\lim_{M\to\infty}\int_0^M e^{-ax}\cos bx\,dx = 0 + \frac{a}{a^2+b^2} \checkmark$$

$$\lim_{M\to\infty}\int_0^M e^{-ax}\sin bx\,dx = \lim_{M\to\infty}\left\{e^{-aM}\left[\frac{-a\sin(bM) - b\cos(bM)}{a^2+b^2}\right] - e^{-a0}\left[\frac{-a\cdot 0 - b\cdot 1}{a^2+b^2}\right]\right.$$
$$= \frac{b}{a^2+b^2} \checkmark$$

Method 3 – Educated Guess / Check by $Ce^{-ax}\cos bx + De^{-ax}\sin bx$

for antiderivatives of both $\to$ deduce formulas above.

$$\#77 \quad F(s) = \int_0^\infty e^{at}\cdot e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

converges for $s > a$.

$$\lim_{T\to+\infty}\int_0^T e^{-(s-a)t}\,dt = \lim_{T\to\infty}\left[-\frac{1}{(s-a)}e^{-(s-a)T} + \frac{1}{s-a}\right] = \frac{1}{s-a}$$

#78 $F(s) = \int_0^\infty t\, e^{-st}\, dt$ converges for $s > 0$ and we find

$$\lim_{T\to\infty} \int_0^T t e^{-st}\, dt = \lim_{T\to\infty} \left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^T$$

? guess/check or by parts

$$= \lim_{T\to\infty} \left[\frac{T e^{-sT}}{-s} - \frac{e^{-sT}}{s^2} + \frac{0}{-s} + \frac{1}{s^2} \right]$$

$= \frac{1}{s^2}$ since $T e^{-sT} \to 0$ (L'Hopital)

COMMON ISSUES ON YOUR WORK

1. $\int \frac{x}{(x+2)^2}\, dx$ – could mess up – partial fractions / substitute

2. $\int x e^{-x^2}\, dx$ – not by parts, but "u-substitution"

3. A few off statements on comparison idea.

Will discuss Challenge after exam.