MATH 116 – HOMEWORK SOLUTION – Integration 7.1 – 7.3

**7.1** **#13** $\int x^2 e^{4x}\,dx$ let $u = x^2$ so $du = 2x\,dx$
and $dv = e^{4x}dx$ so $v = \frac{e^{4x}}{4}$

$= \frac{x^2e^{4x}}{4} - \int \frac{2xe^{4x}}{4}\,dx$ ← Using "by parts" once

↑ then again with $u = x$, $du = dx$
$dv = e^{4x}dx \to v = \frac{e^{4x}}{4}$

$= \frac{x^2e^{4x}}{4} - \frac{xe^{4x}}{8} + \int \frac{e^{4x}}{8}\,dx$

$= \frac{x^2e^{4x}}{4} - \frac{xe^{4x}}{8} + \frac{e^{4x}}{32} + C$ – check by product rule – lots of cancellation

Alternative Guess/Check $\left[(ax^2+bx+c)e^{4x}\right]' = x^2e^{4x}$

Find same values, algebra of product rule.

**#19** $\int \tan^{-1}x\,dx$ : When in doubt, $u = \tan^{-1}x$, $dv = dx$
$du = \frac{1}{1+x^2}dx$, $v = x$

$= x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx = \boxed{x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C}$

**#42** $\int x^n e^{ax}\,dx =$ $u = x^n \to du = nx^{n-1}dx$
$dv = e^{ax}dx \to v = \frac{e^{ax}}{a}$

$\frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$ if $a \neq 0$. [could use in #13 above]

**#63** (a) $\int x f'(x)\,dx = xf(x) - \int f(x)\,dx$ is shown using

$u = x$, $dv = f'(x)\,dx$ : $uv - \int v\,du$ is $xf(x) - \int f(x)dx$
$du = dx$, $v = f(x)$

For $\int xe^{3x}dx$ we get $\frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C$

#64 (a) $y = f^{-1}(x)$, $x = f(y)$ $dx = f'(y)\,dy$

$$\int f^{-1}(x)\,dx = \int \underset{f(x)}{y} \cdot \underbrace{f'(y)\,dy}_{dx}$$ by substitution

(b) From #63 (ing) $\int f^{-1}(x)\,dx = y \cdot f(y) - \int f(y)\,dy$

$= y \cdot x - \int f(y)\,dy$

(c) $y = \ln x \to x = e^y$

so $\int \ln x\,dx = x \cdot \ln x - \int e^y\,dy = x\ln x - e^y + C$

$= \boxed{x \ln x - x + C}$

(d) $y = \sin^{-1} x$, $x = \sin y$ : $\int \sin^{-1} x\,dx = x \sin^{-1} x - \int \sin y\,dy$

$= x \sin^{-1} x + \cos y + C$ : $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1-x^2}$

So $\boxed{x \sin^{-1} x + \sqrt{1-x^2} + C}$

(e) $y = \tan^{-1} x$ [problem 19 above] get $x \tan^{-1} x - \int \tan y\,dy$

so $\int \tan^{-1} x\,dx = x \tan^{-1} x - \ln|\sec y| + C$ but $1 + \tan^2 y = \sec^2 y = 1 + x^2$

and $\ln|\sec y| = \frac{1}{2}\ln(\sec^2 y)$

$= \boxed{x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C}$

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7.2 #11 $\int \sin^5 x\,dx$ – substitution $\sin x\,dx = -du \Rightarrow u = +\cos x$.

or v

$= \int \sin^4 x \cdot \sin x\,dx = \int (\sin^2 x)^2 \sin x\,dx = \int (1-u^2)^2 (-du)$

$= \int [-1 + 2u^2 - u^4]\,du = -u + \frac{2}{3}u^3 - \frac{u^5}{5} + C$, $u = +\cos x$

$= \boxed{-\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + C}$

(a)

<u>Alternatives</u> Reduction $\int \sin^5 x\, dx = \dfrac{-\sin^4 x \cos x}{5} + \dfrac{4}{5}\int \sin^3 x\, dx$

$$= \frac{-\sin^4 x \cos x}{5} + \frac{4}{5}\left(-\frac{1}{3}\sin^2 x \cos x + \frac{2}{3}\int \sin x\, dx\right)$$

$$= \boxed{\frac{-\sin^4 x \cos x}{5} - \frac{4}{15}\sin^2 x \cos x - \frac{8}{15}\cos x + C}$$

(b) Euler $\displaystyle\int \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^5 dx = \frac{1}{32 i^5}\int (e^{ix} - e^{-ix})^5\, dx$

$i^2 = -1$ implies $i^4 = 1$ so $i^5 = i$; Binomial theorem yields

$$\frac{1}{32i}\int \left[e^{5ix} - 5e^{3ix} + 10e^{ix} - 10e^{-ix} + 5e^{-3ix} - e^{-5ix}\right] dx$$

$$= \frac{1}{32i}\left[\frac{e^{5ix}}{5i} - \frac{5e^{3ix}}{3i} + \frac{10e^{ix}}{i} + \frac{10e^{-ix}}{i} - \frac{5e^{-3ix}}{3i} + \frac{e^{-5ix}}{5i}\right] + C$$

using $i^2 = -i$, simplify

$$= -\frac{1}{32}\left[\frac{e^{5ix} + e^{-5ix}}{5} - \frac{5}{3}\left(e^{3ix} + e^{-3ix}\right) + 10\left(e^{ix} + e^{-ix}\right)\right] + C$$

$$= \boxed{-\frac{1}{80}\cos(5x) + \frac{5}{48}\cos(3x) - \frac{5}{8}\cos(x) + C}$$

all are equivalent !!

#12 Similar to 11: easier $\displaystyle\int \cos^3 20x\, dx$ ; substitution $dv = \cos 20x\, dx$, $v = \dfrac{\sin 20x}{20}$ or

$$= \frac{1}{20}\int \left(1 - \sin^2(20x)\right) 20\cos(20x)\, dx$$

$u = \sin 20x$ (easier)

$$= \frac{1}{20}\int (1 - u^2)\, du = \frac{1}{20}\left[u - u^3/3\right] + C$$

$du = 20\cos 20x\, dx$

$= \frac{1}{20}\left[\sin(20x) - \frac{\sin^3(20x)}{3}\right] + C$

or use reduction or Euler

#25 $\int \sec^2 x \tan^{\frac{1}{2}} x \, dx$ If $u = \tan x$, $du = \sec^2 x \, dx$

so $\int u^{1/2} du = \frac{2}{3} u^{3/2} + C = \frac{2}{3}(\tan(x))^{3/2} + C$

#29 $\int_0^{\pi/4} \sec^4 \theta \, d\theta$ let $t = \tan\theta$, $dt = \sec^2\theta \, d\theta$

$1+t^2 = 1+\tan^2\theta = \sec^2\theta$

$= \int (1+t^2)\, dt = t + \frac{t^3}{3}$ so get $\tan\theta + \frac{\tan^3\theta}{3}\Big|_0^{\pi/4} = 1 + \frac{1}{3} - 0$

$= \boxed{4/3}$

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$\boxed{7.3}$ #7 $\int_0^{5/2} \frac{dx}{\sqrt{25-x^2}}$ let $x = 5\sin t$, $dx = 5\cos t \, dt$

$= \int \frac{5\cos t \, dt}{\sqrt{25 - 25\sin^2 t}} = \int dt = t = \sin^{-1}\left(\frac{x}{5}\right)$

So $\int_0^{5/2} \frac{dx}{\sqrt{25-x^2}} = \sin^{-1}\left(\frac{x}{5}\right)\Big|_{x=0}^{x=5/2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) = \boxed{\frac{\pi}{6}}$

#17 $\int \frac{dx}{\sqrt{36-x^2}}$ $x = 6\sin t$ get $\int dt$ as above

$= t + C = \boxed{\sin^{-1}\left(\frac{x}{6}\right) + C}$

**#33** $\int \frac{x^2}{(25+x^2)^2}\,dx$ let $x = 5\tan t$, $dx = 5\sec^2 t\,dt$

Get $\int \frac{25\tan^2 t}{[25+25\tan^2 t]^2} \cdot 5\sec^2 t\,dt = \int \frac{25\cdot 5}{(25)^2}\,\frac{\tan^2 t\,\sec^2 t}{\sec^4 t}\,dt$

$= \frac{1}{5}\int \frac{\tan^2 t}{\sec^2 t}\,dt = \frac{1}{5}\int \underbrace{\sin^2 t\,dt}_{\text{after clean up}} = \frac{t}{10} - \frac{\sin(2t)}{20} + C$

Now $\frac{\sin(2t)}{20} = \frac{2\sin t\cos t}{20} = \frac{2\tan t\cdot\cos^2 t}{20} = \frac{2\tan t}{20\sec^2 t} = \frac{2}{20}\cdot\frac{(x/5)}{[1+x^2/25]}$

$= \frac{2}{20}\cdot 5\cdot\frac{x}{25+x^2} = \frac{1}{2}\frac{x}{25+x^2}$ so get $\int \frac{x^2}{(25+x^2)^2}\,dx = \boxed{\frac{1}{10}\tan^{-1}\left(\frac{x}{5}\right) - \frac{1}{2}\frac{x}{(25+x^2)} + C}$

**#56** $y = \frac{b}{a}\sqrt{a^2-x^2}$ so area $= 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2}\,dx$

let $x = a\sin t$, $a^2 - x^2 = a^2\cos^2 t$ so have

Area $= \frac{b}{a}\cdot 2\cdot \int_{t=-\pi/2}^{t=\pi/2} a^2\cos^2 t\,dt$ : using Frank idea, the integrand like $\frac{1}{2}a^2$

Get $\frac{2b}{a}\cdot a^2\cdot\pi\cdot\frac{1}{2} = \boxed{\pi ab}$