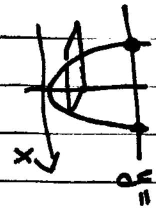
Math 116 Homework 2 Solutions – Prof. Sachs

6.3 **#10** $y=x^2$, $y=1$



At y: cross section is square, base goes from $x_{Left}=-\sqrt{y}$ to $x_{Right}=+\sqrt{y}$

area is $(2\sqrt{y})^2 = 4y$

$$\int_0^1 4y\,dy = 2y^2\Big|_0^1 = 2$$

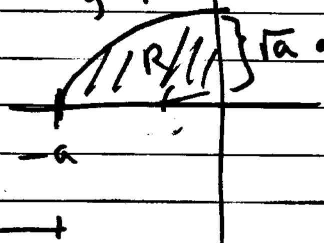
#17 Slice is circular (disk) – area $=\pi\cdot(\text{radius})^2$; radius is e^{-x}

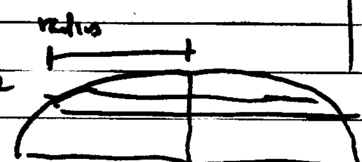
so $$\int_0^{\ln 4} \pi\cdot(e^{-x})^2\,dx = \pi\int_0^{\ln 4} e^{-2x}\,dx = \pi\left[-\frac{e^{-2x}}{2}\right]_{x=0}^{x=\ln 4}$$

$$=\pi/2\left[-e^{-2\ln 4}+e^0\right] = \pi/2\left[-\frac{1}{16}+1\right] = \boxed{\frac{15\pi}{32}}$$

Note $e^{-2\ln 4} = e^{-\ln 16} = \frac{1}{16}$

#49





$x+a=y^2$ so $x=-a+y^2<0$

so $$V=\int_0^{\sqrt{a}} \pi\left[a-y^2\right]^2 dy$$

$$=\pi\cdot\int_0^{\sqrt{a}}\left[a^2-2ay^2+y^4\right]dy$$

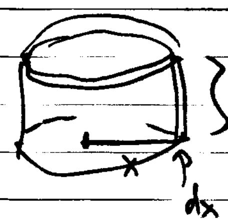
$$=\pi\left[a^2y-\frac{2ay^3}{3}+y^5/5\right]_{y=0}^{y=\sqrt{a}}$$

$$=\pi a^2\cdot a^{1/2}\left[1-2/3+1/5\right] = \pi a^{5/2}\left(\frac{1}{3}+\frac{1}{5}\right)$$

$$=\pi\cdot a^{5/2}\left[\frac{8}{15}\right]$$. Compared to cone

cone $\frac{1}{3}\pi a^2\cdot a^{1/2}$: Factor is $\boxed{8/5}$

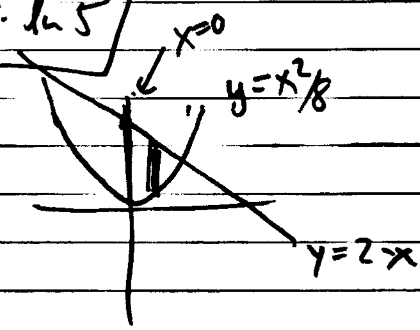
6.4 **#5** around y axis



$$\int_0^2 2\pi x\cdot\frac{1}{1+x^2}\,dx$$

u substitution: $u = 1+x^2$, $du = 2x\,dx$

$$= \pi \cdot \ln(1+x^2)\Big|_{x=0}^{x=2} = \pi[\ln 5 - \ln 1] = \boxed{\pi \cdot \ln 5}$$



#28 $y = x^2/8$, $y = 2-x$, $x = 0$ about y axis

Intersection $x^2/8 = 2-x$ or $x^2 = 16-8x$

$$x^2 + 8x + 16 = 32$$

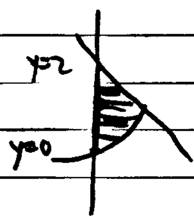
perfect square so $(x+4)^2 = (4\sqrt{2})^2$

$x = -4 \pm 4\sqrt{2}$ Assume in quadrant I. $\boxed{\text{SHELLS}}$

$$\int_0^{4\sqrt{2}-4} 2\pi x\,[\underset{\text{top}}{2} - x - \underset{\text{bottom}}{x^2/8}]\,dx = 2\pi\int_0^{4\sqrt{2}-4}[2x - x^2 - x^3/8]\,dx$$

$$= 2\pi \cdot \left[x^2 - x^3/3 - x^4/32\right]_0^{4\sqrt{2}-4} =$$

By $\boxed{\text{DISKS}}$



intersection $x = -4+4\sqrt{2}$ has $y = 2-x = 2+4-4\sqrt{2}$
$= 6-4\sqrt{2}$

bottom "half": $x^2/8 = y$ so $x^2 = 8y$ — area of disk $\pi x^2 = \pi \cdot 8y$

top: $y = 2-x$, $x = 2-y$

$$\int_0^{6-4\sqrt{2}} \pi \cdot [8y]\,dy + \int_{6-4\sqrt{2}}^{2} \pi[2-y]^2\,dy$$

$$= \pi\left[4y^2\right]_0^{6-4\sqrt{2}} + \pi\Big[\underset{\text{shek}}{-}\frac{(2-y)^3}{3}\Big]_{6-4\sqrt{2}}^{2} = \pi \cdot 4 \cdot (6-4\sqrt{2})^2 + \pi\frac{(4\sqrt{2}-4)^3}{3}$$

$$=$$

#50 Spherical cap — enough to do for $0 \le h \le r$ by symmetry

(a) washers: top slice $z = r$, bottom $z = r-h$: at height z have $x^2+y^2 \le \underset{\text{radius squared}}{r^2 - z^2}$

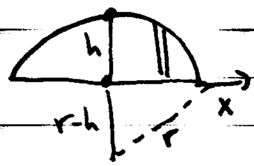
so $$\int_{r-h}^{r} \pi(r^2 - z^2)\,dz = \pi\left[r^2 z - z^3/3\right]_{r-h}^{r}$$

$$= \pi\left[r^3 - r^3/3 - r^2(r-h) + \frac{(r-h)^3}{3}\right] = \pi\left(r^3 - \frac{r^3}{3} - r^3 + hr^2 + \frac{r^3 - 3r^2h + 3rh^2}{3} - h^3/3\right)$$

$$= \pi\left(hr^2 - r^2h + rh^2 - h^3/3\right) = \boxed{\pi h^2 (r - h/3)}$$

ok for $h=0$, $h=r$
nothing [hemisphere]

(b) Spin



top $\sqrt{r^2 - x^2}$, bottom $r-h$ Domain $0 \le x \le \sqrt{r^2 - (r-h)^2} = \sqrt{2rh - h^2}$

$$\int_0^{\sqrt{2rh-h^2}} 2\pi x \cdot \left(\sqrt{r^2 - x^2} - (r-h)\right) dx = 2\pi\left[\frac{(r^2-x^2)^{3/2}}{-3} - \frac{x^2}{2}(r-h)\right]_0^{\sqrt{2rh-h^2}}$$

$$= 2\pi\left[-\frac{1}{3}\left(r^2 - (2rh-h^2)\right)^{3/2} + \frac{1}{3}r^3 - \frac{1}{2}(2rh-h^2)(r-h) + 0\right]$$

$$= 2\pi\left[-\frac{1}{3}(r-h)^3 + \frac{1}{3}r^3 - \frac{1}{2}(2rh-h^2)(r-h)\right]$$

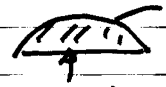
$$= \frac{2\pi}{3}\left[-(r^3 - 3r^2h + 3rh^2 - h^3) + r^3\right] - \pi\left[2r^2h - h^2r - 2rh^2 + h^3\right]$$

$$= \underbrace{\frac{2\pi}{3}\left[3r^2h - 3rh^2 + h^3\right] - 2\pi r^2 h}_{\text{cancel}} + 3\pi h^2 r - \pi h^3$$

$$= \pi \cdot h^2 \left[r - h/3\right] \text{ as above!}$$

(c) ~~Not sure what they mean~~: ~~slice in x - same as in (b) but doubled~~

At x!



$z = r-h$

$y^2 + z^2 \le r^2 - x^2 \to$ gets messy fast: slice in $x \to$ cross section area via y-z integral in y or one in z

top: $z = \sqrt{r^2 - x^2 - y^2}$
bottom $z = r-h$

$\boxed{\text{EULER'S FORMULA}}$ $e^{3it} = (e^{it})^3$ becomes

$$\cos 3t + i\sin 3t = (\cos t + i\sin t)^3 = \cos^3 t + 3i\cos^2 t\sin t + 3\cdot i^2\cos t\sin^2 t + i^3\sin^3 t$$

$$= \cos^3 t - 3\cos t\sin^2 t + i(3\cos^2 t\sin t - \sin^3 t)$$

Easy! versus $\cos(3t) = \cos(t+2t) = \cos t\cos 2t - \sin t\sin 2t$

$$= \cos t[\cos^2 t - \sin^2 t] - \sin t[2\sin t\cos t]$$

$$= \cos^3 t - \cos t\sin^2 t - 2\sin^2 t\cos t$$

$$= \cos^3 t - 3\sin^2 t\cos t \quad \text{SAME!}$$

and $\sin(3t) = \sin(t+2t) = \sin t\cos(2t) + \sin(2t)\cos t$

$$= \sin t[\cos^2 t - \sin^2 t] + 2\sin t\cos^2 t = 3\sin t\cos^2 t - \sin^3 t.$$

SAME!

From my personal perspective, Euler's formula version gave __both__ results in a lot less time / less thought.