MATH 114 – HOMEWORK 4 SOLUTIONS – Prof SACHS

7.7 #54: $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln(x)}{x^2}\,dx$

Let $u = \ln(x)$, $dv = \frac{1}{x^2}\,dx$
so $du = 1/x\,dx$, $v = -\frac{1}{x}$

$$: \int_1^b \frac{\ln(x)}{x^2}\,dx = -\frac{\ln(b)}{b} - \int_1^b -\frac{1}{x}\cdot\frac{1}{x}\,dx$$

$$= -\frac{\ln(b)}{b} - \frac{1}{x}\Big|_1^b$$

$$= \frac{\ln(b)}{b} - \frac{1}{b} + 1$$

As $b \to +\infty$, $\frac{\ln(b)}{b} \to 0$ (L'Hopital $\displaystyle\lim_{b\to\infty}\frac{\ln b}{b} = \lim_{b\to\infty}\frac{1/b}{1} = 0$)

So we get $\boxed{1}$

#70 (a) $\displaystyle 0.0005\int_{15,000}^\infty e^{-0.00005t}\,dt = \lim_{b\to\infty} 0.00005\int_{15,000}^b e^{-0.00005t}\,dt$

$$= \lim_{b\to\infty}\left[-e^{-0.00005t}\right]_{15,000}^b = +e^{-0.00005\cdot 15,000} \approx 0.472$$

(b) lasts at least 30,000 hrs, is similar to (a), prob is $e^{-0.00005\cdot 30,000}$

which is the square of above so ratio: 0.472

(c) 1: this means all work at time 0

#71 $\displaystyle 0.00005\int_0^\infty t e^{-0.00005t}\,dt = 0.00005\lim_{b\to\infty}\int_0^b te^{-0.00005t}\,dt$

$$= 0.00005\cdot\lim_{b\to\infty}\left[-\frac{e^{-0.00005t}\cdot(0.00005t+1)}{(0.00005)^2}\right]_0^b = \frac{1}{0.00005}$$

$= 20,000$ hrs.

#73 (a) $W = \int_R^\infty \frac{GMm}{x^2}\,dx = \lim_{b\to\infty} \int_R^b \frac{GMm}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{GMm}{b} + \frac{GMm}{R}\right]$

$= \frac{GMm}{R} = \left(\frac{4\times10^{14}\,m^3/s^3}{6.370\times10^3\,m}\right)m \approx \left[6.28\times10^7\,J\right]m$

(b) $\frac{1}{2}mv_e^2 = 6.28\times10^7\,J$, $v_e \approx 11.2$ km/sec

(c) $\frac{GM}{R} \geq \frac{1}{2}c^2$ so $R \leq \frac{2GM}{c^2} \approx 9$ mm

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7.8 #13 $y' = 3y - 4 = 3(y - 4/3)$ Book $\frac{dy}{3y-4} = dt$

so $\ln|3y-4| - \ln|3y_0-4| = 3t$ so $\left|\frac{3y-4}{3y_0-4}\right| = e^{3t}$

$3y - 4 = \pm(3y_0-4)e^{3t}$ let $3C = \pm(3y_0-4)$

$3y = 4 + 3Ce^{3t}$, $\boxed{y = 4/3 + Ce^{3t}}$

My way – let $u = y - 4/3$, $\frac{du}{dt} = \frac{dy}{dt} = 3u$ so $u = Ce^{3t}$

$y = 4/3 + Ce^{3t}$

#18 $\frac{dy}{dx} = -y + 2$, $y(0) = -2$ let $u = y - 2$, $\frac{du}{dx} = -u$

$u = Ce^{-x}$

$y = 2 + Ce^{-x}$: at $x=0$, $y=-2$ so $C = -4$

$\boxed{y = 2 - 4e^{-x}}$

#54 (a) $m\frac{dv}{dt} = mg - kv^2$ so $\frac{dv}{dt} = g - av^2$, $a = k/m$

(b) When $g - av^2 = 0$ we get $v' = 0$ — $v_\tau = (g/a)^{1/2}$

$\int \frac{dv}{g-av^2} = \int dt$ so $-\frac{1}{a}\int \frac{dv}{v^2-v_*^2} = t + A$ - const

By partial fracts $-\frac{1}{2av_*} \ln\left|\frac{v-v_*}{v+v_*}\right| = t+A$

so $\frac{v-v_*}{v+v_*} = C e^{-2av_* t}$ At $t=0$, $v=0$ so $C=-1$

$v - v_* = -(v+v_*)e^{-2av_* t}$ solve for v: $\boxed{v = \frac{1-e^{-2av_* t}}{1+e^{-2av_* t}} v_*}$

As $t \to \infty$ get v_* ✓

$\boxed{\#55}$ $v' = g - bv$, $b = k/m$ $\quad v_* = g/b$

$\frac{dv}{g-bv} = dt$ or let $u = v - g/b$, $\frac{du}{dt} = \frac{dv}{dt} = -bu$

$u = Ce^{-bt}$, $v = g/b - Ce^{-bt}$

$t=0$: $C = g/b = v_*$ $\quad \boxed{v = v_*\left(1-e^{-bt}\right)}$

as $t \to \infty$ get v_* ✓