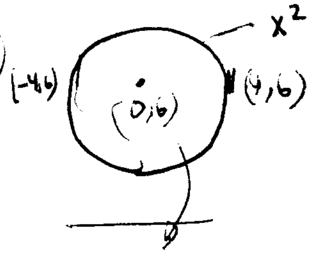
MATH 114 – Prof. Sacks – Solutions to HOMEWORK 3

7.3 #69

$x^2+(y-6)^2=4^2$



Washers Top. $y-6=\sqrt{16-x^2}$

Bottom $y-6=-\sqrt{16-x^2}$

x-range: $-4 \le x \le 4$

$$\int_{-4}^{4}\pi\left[\left(6+\sqrt{16-x^2}\right)^2-\left(6-\sqrt{16-x^2}\right)^2\right]dx = \int_{-4}^{4} 24\pi\cdot\sqrt{16-x^2}\,dx$$

Either recognize as area of semicircle or do trig substitution $x=4\sin t$, $dx=4\cos t\,dt$

get answer $48\pi\cdot\frac{\pi}{4}\cdot 4^2 = 192\pi^2$.

Note: Famous theorem of Pappus – area of face • distance of "center of mass"

Area of face: 16π, rotated by $2\pi\cdot 6$

#72 As done in class: $\frac{\sin\theta}{r^2}=\frac{a}{r^3}=\frac{a}{(a^2+y^2)^{3/2}}$, so we need

(a) $\int_{-L}^{L}\frac{a}{(a^2+y^2)^{3/2}}\,dy$ to find this, use trig substitution $y=a\tan t$

$dy=a\sec^2 t\,dt$

$a^2+y^2=a^2\sec^2 t$

$$\int\frac{a}{(a^2+y^2)^{3/2}}\,dy=\int\frac{a\cdot a\sec^2 t}{a^3\sec^3 t}\,dt=\frac{1}{a}\int\cos t\,dt=\frac{1}{a}\sin t=\frac{1}{a}\cdot\frac{y}{(a^2+y^2)^{1/2}}+C$$

so we get $\int_{-L}^{L}\frac{a}{(a^2+y^2)^{3/2}}\,dy=\frac{1}{a}\frac{y}{(a^2+y^2)^{1/2}}\Big|_{-L}^{L}=\frac{2}{a}\cdot\frac{L}{(a^2+L^2)^{1/2}}$

Thus $$B(a)=\frac{\mu_0 I}{4\pi}\cdot\frac{2L}{a(a^2+L^2)^{1/2}}=\frac{\mu_0 I L}{2\pi a\sqrt{a^2+L^2}}$$

(b) As $L \to +\infty$, $\frac{L}{\sqrt{a^2+L^2}} = \frac{1}{\sqrt{1+a^2/L^2}} \to 1$ so we get $\boxed{\frac{\mu_0 I}{2\pi a}}$

#73 (a) $\int_a^b \sqrt{\frac{1-\cos t}{g(\cos a - \cos t)}}\, dt$ using $u = \cos t$: u runs from $\cos a$ to $\cos b$

$du = \sin t\, dt = \sqrt{1-u^2}\, dt$

so $dt = \frac{du}{\sqrt{1-u^2}}$

we get $\int_{\cos a}^{\cos b} \sqrt{\frac{1-u}{g(\cos a - u)}} \cdot \frac{1}{\sqrt{1-u^2}}\, du$

$= \int_{\cos a}^{\cos b} \frac{1}{\sqrt{g(\cos a - u)(1+u)}}\, du$ this is done by completing the square and another substitution

$(1+u)(\cos a - u) = \cos a + (\cos a - 1)u - u^2$ needs $\left(\frac{\cos a - 1}{2}\right)^2$ fix

leave it for now as let $k^2 = \cos a + \left(\frac{\cos a - 1}{2}\right)^2 = \left(\frac{\cos a + 1}{2}\right)^2$, $v = u - \left(\frac{\cos a - 1}{2}\right)$

$\int \frac{1}{\sqrt{k^2 - v^2}}\, dv = \sin^{-1}(v/k) + C$ — messy with endpoints.

(b) when $b = \pi$, $\cos \pi = -1$ and we have

$+\frac{1}{\sqrt{g}}\left[\sin^{-1}\left(\frac{\cos a + 1}{\cos a + 1}\right) - \sin^{-1}\left(\frac{-2 - \cos a + 1}{\cos a + 1}\right)\right] = \frac{1}{\sqrt{g}}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$

$= \pi/\sqrt{g}$ whew!

$\boxed{7.4}$ #8 $\frac{x^2 - 3x}{x^3 - 3x^2 - 4x}$ can divide by x (as noted in class)

$= \frac{x-3}{x^2-3x-4} = \frac{A}{(x-4)} + \frac{B}{x+1}$ using limits ("cover up") or expanding/solving, find $A = 1/5$

$B = 4/5$

#26 $\dfrac{2}{x(x^2-6x+9)} = \dfrac{2}{x(x-3)^2} = \dfrac{A}{x} + \dfrac{B}{(x-3)^2} + \dfrac{C}{(x-3)}$ (your A, B, C can be different places)

#29 $\dfrac{2x^2+3}{(x^2-8x+16)(x^2+3x+4)} = \dfrac{A}{(x-4)^2} + \dfrac{B}{(x-4)} + \dfrac{Cx+D}{x^2+3x+4}$

#81 $x^4(1-x)^4 = x^4(1-4x+6x^2-4x^3+x^4)$

$= x^4-4x^5+6x^6-4x^7+x^8$

so $\dfrac{x^4(1-x)^4}{1+x^2} = \dfrac{x^8-4x^7+6x^6-4x^5+x^4}{x^2+1} = x^6-4x^5+5x^4-4x^2+4-\dfrac{4}{1+x^2}$

Integral is then $\displaystyle\int_0^1 \left[x^6-4x^5+5x^4-4x^3+4-\frac{4}{1+x^2}\right]dx$

$= \frac{1}{7}-\frac{2}{3}+1-\frac{4}{3}+4-\pi = \frac{22}{7}-\pi$. Since $\dfrac{x^4(1-x)^4}{1+x^2} > 0$ except at endpoints

$\frac{22}{7}-\pi > 0$. cute!?

9.1 **#8** $f(x) = x^{1/2}$ at $a=4$: $f(4)=2$, $f'(x) = \frac{1}{2}\cdot x^{-1/2}$

$f''(x) = -\frac{1}{4}\cdot x^{-3/2}$

so $f'(4) = \frac{1}{4}$, $f''(4) = -\frac{1}{32}$

(a) $\underbrace{f(x) \approx 2 + \frac{1}{4}(x-4)}_{p_1(x)}$, $f(3.9) \approx \underbrace{2+\frac{1}{4}(-0.1)}_{p_1(3.9)} = 1.975$ ✓(c)

(b) $f(x) \approx 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 \leftarrow p_2(x)$; $p_2(3.9) = 1.97484375$ (c)

#23 $\dfrac{1}{\sqrt{1.08}}$: $f(x) = \dfrac{1}{\sqrt{1+x}}$, $p_2(x) = 1 - x/2 + 3x^2/8$

$p_2(0.08) = 0.9624$, error $\approx 1.50 \times 10^{-4}$ (using calculator/computer for $f(0.08)$)

#27 $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$ – all values $\pm \frac{1}{\sqrt{2}}$

$p_0(x) = 1/\sqrt{2}$, $p_1(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \pi/4)$, $p_2(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \pi/4) - \frac{1}{2\sqrt{2}}(x - \pi/4)^2$

#67 Matches are (a) – C, (b) – E, (c) – A, (d) – D, (e) – B, (f) – F

Reasons.
$(1+2x)^{1/2} \Rightarrow 1 + x + \dots$ only C
$(1+2x)^{-1/2} \Rightarrow 1 - x + \dots$ only E
$e^{2x} \to 1 + 2x + \dots$ only A
$(1+2x)^{-1} \to 1 - 2x + 4x^2 \to$ D
$(1+2x)^{-3} \to 1 - 6x + \dots$ only B
$e^{-2x} \to 1 - 2x + 2x^2 \to$ F

SADLY, didn't need second derivatives for most of them only D vs F

#84 (a) $f(x) = f(a) + \int_a^x f'(t)\,dt$ – automatic (FTC)

(b) using hint, $f(x) = f(a) + f'(t)(t-x)\Big|_a^x - \int_a^x (t-x) f''(t)\,dt$
$= f(a) + f'(a)(x-a) + \int_a^x (x-t) f''(t)\,dt$

(c) Continuing get $f(x) = p_n(x) + \int_a^x \dfrac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$: make u-sub. $du = \dfrac{(x-t)^n}{n!}\,dt$

to get (d) by MVT for integrals.