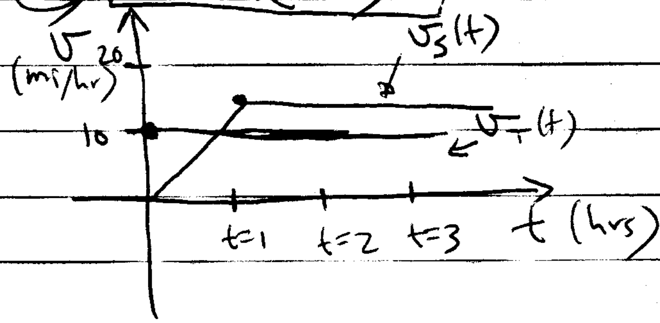
HOMEWORK # 2 SOLUTION – MATH 114 Prof. SACHS

(1) Paragraph you wrote

(2) #47 (6.1)



(a)

(b) From area: Theo goes 10 miles, Sasha goes $\frac{1}{2}\cdot 1\cdot 15 = 7.5$ mi
triangle
Theo goes faster in 1 hr

(c) Add to (b) the second hour: Theo total is $10+10=20$ mi
Sasha total $7.5+15 = 22.5$ mi

Sasha goes farther in 2 hours – area is bigger

(d) 10 mi – Theo: 1 hr, Sasha clearly more time (7.5 in 1st)
Theo first at 10 mi; 15 miles – Theo $1\frac{1}{2}$ hr (base of rect, area 15)
Sasha 7.5 in 1st hr, another 7.5 in $\frac{1}{2}$ hr.
so ties Theo
20 mi – Sasha is first (from 15 mi mark, going faster)

(e) Theo goes 19.8 mi takes $19.8/10 = 1.98$ hrs.

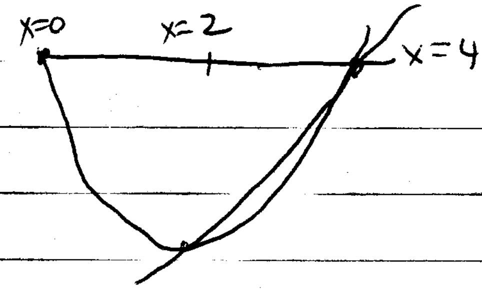
Sasha 15 miles in 1.5 hrs, next 5 miles in $5/15 = \frac{1}{3}$ hr

So Sasha gets there first. $(1.5 + 1/3 = 3/2 + 1/3 = 11/6 \text{ hr})$

(f) Theo gets 0.2 hr headstart – 2 miles so in race to finish

needs 18 miles more, 1.8 hrs to 70 mile mark
Sasha uses $11/6$ hr. $\approx 1.83\overline{3}$ so Theo wins.

(3) 6.2 # 28 (as in class)



Intersection pts - line with parabola

$$x^2-4x = 2x-8 = 2(x-4) \qquad x=4 \text{ or } x=2$$
$$x(x-4)$$

Area as x-integral [top: $y=0$, bottom $y=x^2-4x$ $(0\le x\le 2)$
and $y=2x-8$ $(2\le x\le 4)$]

so $\boxed{\text{AREA} = \int_{x=0}^{x=2}\left[0-(x^2-4x)\right]dx + \int_{x=2}^{x=4}\left[0-(2x-8)\right]dx}$

For y-integral [right $y=2x-8$ so $y+8=2x$, $x=\frac{y+8}{2}$ (or $\frac{y}{2}+4$)]

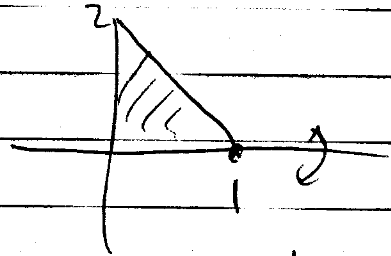
left $y=x^2-4x$ so $y+4=x^2-4x+4 = (x-2)^2$

$\pm\sqrt{y+4} = x-2$ so $x = 2\pm\sqrt{y+4}$ — Pick − since left edge is left of center symmetry line $x=2$

ANS (note $x=2 \Leftrightarrow y=-4$)

$$\int_{y=-4}^{y=0}\left[\frac{y+8}{2} - \left(2-\sqrt{y+4}\right)\right]dy$$

(4) #16, Sect 6.3 $\quad y = 2-2x,\ y=0,\ x=0$



$$\int_{x=0}^{x=1} \pi\cdot(2-2x)^2\,dx$$

Either expand: $\displaystyle\int_0^1 \pi\left[4-8x+4x^2\right]dx = \pi\cdot\left[4x-4x^2+\tfrac{4}{3}x^3\right]_0^1$

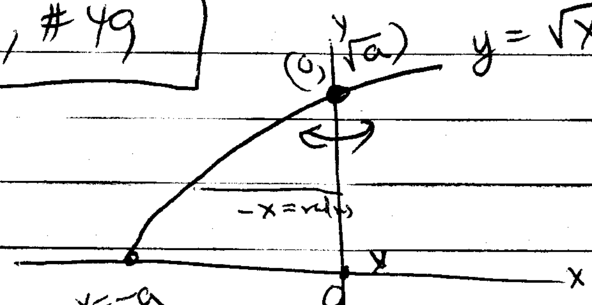
$= \boxed{4/3\,\pi} \quad \left[\tfrac{1}{3}\pi r^2\cdot h\right]$ for cone

OR $\displaystyle\int_0^1 \pi(2-2x)^2\,dx \quad u = 2-2x,\ du = -2\,dx$ so $dx = -\tfrac{1}{2}\,du$

Note $\displaystyle\int \pi\cdot(u)^2\cdot\left(-\tfrac{1}{2}du\right) = -\frac{\pi}{2}\cdot u^3/3$

So $\displaystyle\int_0^1 \pi(2-2x)^2\,dx = -\frac{\pi(2-2x)^3}{6}\Bigg|_{x=0}^{x=1} = \pi\cdot\frac{8}{6} = \boxed{\frac{4\pi}{3}}$

(5) 6.3, #49 $\quad y = \sqrt{x+a}$



$y = \sqrt{x+a}$

$y^2 = x+a$

$x = y^2 - a$

$$\int_{y=0}^{y=\sqrt{a}} \pi(-x)^2\,dy = \pi\int_{y=0}^{y=\sqrt{a}} (y^2-a)^2\,dy = \pi\int_0^{\sqrt{a}} (y^4-2y^2a+a^2)\,dy$$

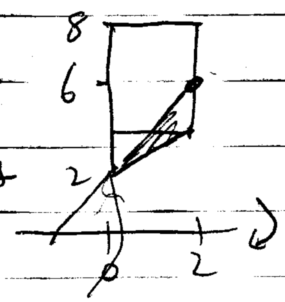
$$= \pi\left[y^5/5 - \tfrac{2}{3}y^3a + a^2\cdot y\right]_{y=0}^{y=\sqrt{a}} = \pi a^{5/2}\left[\tfrac{1}{5} - \tfrac{2}{3} + 1\right]$$

$$= \pi \cdot a^{5/2}\left[\frac{1}{5}+\frac{1}{3}\right] = \pi a^{5/2}\cdot 8/15$$

Cone: Base area πa^2, height $a^{1/2}$ Volume $\pi a^{5/2}\cdot \frac{1}{3}$

Ratio is $8/5$ — just like Fermat!

(6) $\boxed{6.4, \#12}$ $y=8,\ y=2x+2,\ x=0,\ x=2$



Shell radius: y

Shell height: Rt − Left

Left is $x=0$ always

Rt is $x=2$, $6 \le y \le 8$

$$x = \frac{y-2}{2} = \frac{y}{2}-1$$

Volume is $\displaystyle\int_{y=2}^{y=6} 2\pi y\left[\frac{y-2}{2}\right]dy$

$$+\int_{y=6}^{y=8} 2\pi y\cdot 2\,dy$$

$$= \int_2^6 \pi\cdot(y^2-2y)\,dy + \int_6^8 4\pi y\,dy = \pi\left[y^3/3 - y^2\right]_2^6$$

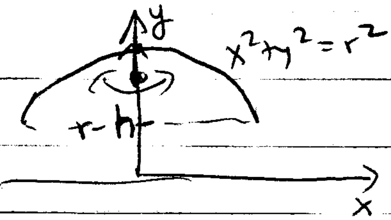
$$+ 2\pi y^2\Big|_6^8$$

$$= \pi\cdot\left[6^3/3 - 6^2 - \frac{8}{3} + 4 + 2\cdot 64 - 2\cdot 36\right]$$

$$= \pi\left[\cancel{72} - 36 - 8/3 + 4 + 128 - \cancel{72}\right]$$

$$= \pi\left[96 - 8/3\right] = \pi\cdot\left(\frac{280}{3}\right)$$ — COULD DO AS CONE VOLUME LESS OTHER CONE REMOVED

(7) 6.4, #50



(a) Washer/disk: Slice in y: $\int_{y=r-h}^{y=r} \pi \cdot x^2\,dy = \int_{r-h}^{r} \pi\cdot[r^2-y^2]\,dy$

$= \pi\cdot\left[r^2y - y^3/3\right]\Big|_{r-h}^{r} = \pi\left[r^2(r) - r^3/3 - r^2(r-h) + \frac{(r-h)^3}{3}\right]$

Some algebra helps $r^2\cdot r - r^2(r-h) = r^2h$

$\frac{(r-h)^3}{3} - \frac{r^3}{3} = -\frac{1}{3}\cdot h\left[(r-h)^2+(r-h)r+r^2\right]$

diff of cubes or expand

Volume is $\frac{\pi\cdot h}{3}\left[3r^2 - r^2 - r^2 + hr - r^2 + 2rh - h^2\right]$

$= \frac{\pi h}{3}\left[3rh - h^2\right] = \frac{\pi h^2}{3}(3r-h)$. $h=0$ get 0 ✓
$h=r$ get $\frac{2}{3}\pi r^3$ ✓

(b) Shell — Slice in x: $0 \le x \le \sqrt{r^2-(r-h)^2}$

Shell radius: x, height: top $y=\sqrt{r^2-x^2}$ to bottom $y=r-h$

$$\int_0^{\sqrt{r^2-(r-h)^2}} 2\pi x\cdot\left[\sqrt{r^2-x^2} - (r-h)\right]dx$$

$$= \pi\left[-\frac{(r^2-x^2)^{3/2}}{3/2} - (r-h)\cdot x^2\right]_0^{\sqrt{r^2-(r-h)^2}}$$

← as in sphere

$$= \pi\left[-\frac{\left(r^2-(r^2-(r-h)^2)\right)^{3/2}}{3/2} + r^3/\tfrac{3}{2} - (r-h)\left(r^2-(r-h)^2\right)\right]$$

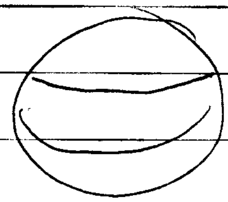
(More algebra) $= \pi\left[\left(\frac{2}{3}\left(r^3-(r-h)^3\right)-(r-h)(2rh-h^2)\right)\right]$

$= \frac{\pi\cdot h}{3}\left[2\left(r^2+r(r-h)+(r-h)^2\right)-3(r-h)(2r-h)\right]$

$= \frac{\pi h}{3}\left[2\left(r^2+r^2-rh+r^2-2rh+h^2\right)-6r^2+9rh-3h^2\right]$

$= \frac{\pi h}{3}\left[3rh-h^2\right] = \frac{\pi h^2}{3}(3r-h)$ ✓

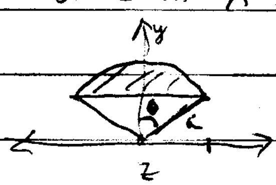
(c) Slicing $x^2+y^2+z^2\le a^2$, $r-h\le y\le r$



x slices — while before $0\le x\le\sqrt{r^2-(r-h)^2}$

now $-\sqrt{r^2-(r-h)^2}\le x\le\sqrt{r^2-(r-h)^2}$

Slice at x : $y^2+z^2\le a^2$, $r-h\le y\le\sqrt{r^2-x^2}$



To be computed : Area $A(x)$ — getting harder than (a) or (b)

$a^2=r^2-x^2$

At corner : $r-h=\sqrt{r^2-x^2}=y$

Area of slice at x is $\pi\cdot(r^2-x^2)\cdot\frac{\theta}{2\pi}-\sqrt{r^2-x^2-(r-h)^2}\,(r-h)$

fraction of circle

z range ($\frac{1}{2}$ b·h) = yz wedge

$\cos\theta=\frac{r-h}{\sqrt{r^2-x^2}}$ so end up with horrible integral

$$\int_{-\sqrt{r^2-(r-h)^2}}^{\sqrt{r^2-(r-h)^2}}\left[\frac{1}{2}(r^2-x^2)\cos^{-1}\left(\frac{r-h}{\sqrt{r^2-x^2}}\right)-(r-h)\sqrt{r^2-(r-h)^2-x^2}\right]dx$$