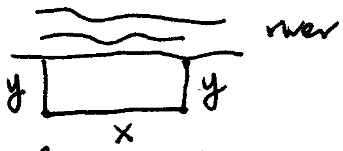
MATH 113 – HOMEWORK 8 SOLUTION – Prof. SACHS

3.5

#7



LARGEST area with 800 m of wire

Let x = length ∥ to river, y = length ⊥ to river. Total length $L = x + 2y = 800$

Area $= xy$ – eliminate x or y using $L = 800$.

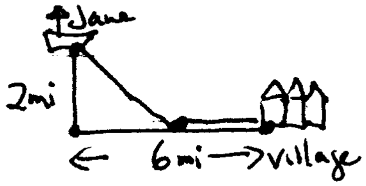
Using x: $A(x) = x \cdot \left(\frac{800-x}{2}\right) = 400x - x^2/2$; in y, get $800y - 2y^2$

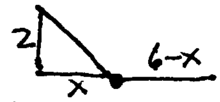
Take derivative to find max. $(A''(x) < 0)$: $400 - x = 0$, $x = 400$

$2y = 400$
$y = 200$

Area $= 200 \times 400 = 80000$ sq m

#32



For path of form 

we want to express total time as function

let x = distance along shore from boat. Then distance over water is $\sqrt{x^2+2^2}$, over land. Note that time = $\frac{\text{distance}}{\text{rate}}$ to get: Total time = time on water + time on land

Land: 5 mph
Water 2 mph

so $T(x) = \frac{\sqrt{x^2+4}}{2} + \frac{6-x}{5}$ → minimized when $\frac{dT}{dx} = 0$

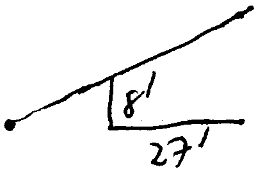
$T'(x) = \underbrace{\frac{1}{2}\frac{1}{2} \cdot (x^2+4)^{-1/2} \cdot 2x}_{\text{chain rule}} - \frac{1}{5}$ so $T' = 0 \Leftrightarrow \frac{x}{2\sqrt{x^2+4}} - \frac{1}{5} = 0$

Squaring to solve we get $\frac{x^2}{4(x^2+4)} = \frac{1}{25}$ so $25x^2 = 4x^2 + 16$

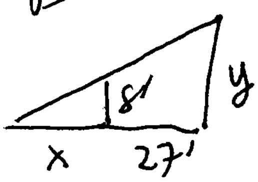
$21x^2 = 16$, $x = \sqrt{\frac{16}{21}}$

$\approx .87287$

#33



Several possible set-ups

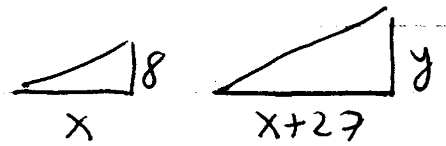


(a) Let x = distance from edge to 8' wall

Beam length is $\sqrt{(x+27)^2 + y^2}$

...OR length is sum of two pieces : $\sqrt{x^2+8^2} + \sqrt{27^2+(y-8)^2}$

Relating x, y uses similar triangles



So $\dfrac{y}{x+27} = 8/x \Rightarrow y = \dfrac{(x+27)\cdot 8}{x} = 8\left(\dfrac{x+27}{x}\right) = 8\left(1+\dfrac{27}{x}\right)$

Therefore first version gives $L(x) = \sqrt{(x+27)^2 + \dfrac{8^2(x+27)^2}{x^2}}$

$$= (x+27)\sqrt{1+64/x^2}$$

using common factor.

Second version $L(x) = \sqrt{x^2+64} + 27\sqrt{1+64/x^2}$

using $y-8 = \dfrac{8\cdot 27}{x}$

Alternate uses trig (see below).

Version 1

We find where $L'(x) = 0 \iff (x+27)'\sqrt{1+64/x^2} + (x+27)\left(\sqrt{1+64/x^2}\right)'$

$$= 1\cdot(1+64/x^2)^{\frac{1}{2}} + (x+27)\cdot\tfrac{1}{2}\cdot\left(1+\frac{64}{x^2}\right)^{-1/2}\left(\frac{-128}{x^3}\right)$$

so $L'(x) = 0 \iff \left(1+\dfrac{64}{x^2}\right)^{1/2} - \dfrac{64\,(x+27)}{x^3\,(1+64/x^2)^{1/2}} = 0$

Cross multiply to get $x^3\left(1+\dfrac{64}{x^2}\right)^{1/2}\left(1+\dfrac{64}{x^2}\right)^{1/2} = 64(x+27)$

or $x^3\left(1+\dfrac{64}{x^2}\right) = 64(x+27) \iff x^3 = 64\cdot 27 \iff x = 4\cdot 3$
$= 12$ ft

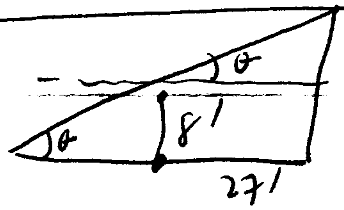
We find $x = 12$, length of beam is $\sqrt{39^2+26^2} = 13\sqrt{13}$
≈ 46.872 ft

Version 2

$L'(x) = \frac{1}{2}(x^2+64)^{-1/2}\cdot 2x + 27\left(1+64/x^2\right)^{-1/2}\cdot\dfrac{(-128)}{x^3}$

Using $(x^2+64)^{-1/2} = \left(x^2(1+64/x^2)\right)^{-1/2} = \frac{1}{x}\cdot(1+64/x^3)$ we get same answer.

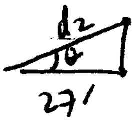
Version 3 - TRIG



let θ = angle of beam

length of beam = sum of two lengths (hypotenuses)

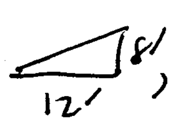
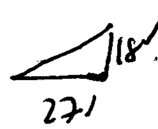
$8/d_1 = \sin\theta$ so $d_1 = 8/\sin\theta = 8\csc\theta$



$27/d_2 = \cos\theta$ so $d_2 = \dfrac{27}{\cos\theta} = 27\sec\theta$

Total length: $L(\theta) = 8\csc\theta + 27\sec\theta$, $\dfrac{dL}{d\theta} = -8\csc\theta\cot\theta + 27\sec\theta\tan\theta$

so $\dfrac{dL}{d\theta} = 0 \Leftrightarrow \dfrac{-8\cos\theta}{\sin^2\theta} + \dfrac{27\sin\theta}{\cos^2\theta} = 0 \Leftrightarrow \dfrac{\sin^3\theta}{\cos^3\theta} = \dfrac{8}{27}$

so $\dfrac{\sin\theta}{\cos\theta} = 2/3$ which allows us to fill in triangles , 

#54 (a) $v = c(r_0 - r)r^2$ cm/sec for $\dfrac{r_0}{2} \le r \le r_0$

r_0, c constants $\rightarrow \dfrac{dv(r)}{dr} = c\cdot(-1)r^2 + c(r_0 - r)\cdot(2r)$

$\uparrow$ product rule / use common

$= cr[-r + 2(r_0 - r)] \leftarrow$ factors

$= cr[2r_0 - 3r]$

So $\dfrac{dv}{dr} = 0 \Leftrightarrow r = 0$ or $3r = 2r_0 \Leftrightarrow r = \frac{2}{3}r_0$

Since $\dfrac{dv}{dr}$ goes from positive to negative, this is a local max at $r = 2/3\, r_0$