Math 113 – Exam 3 – Fall 2004 – Professor Sachs

Name: SOLUTION

Answer each question on this paper. Read them carefully and answer clearly. Exam ends at 2:20pm. The GMU Honor Code is in effect. The exam is worth 100 points.

1. (10 points) Find the absolute extreme values of the function $f(x) = x^3 - 3x$ on the interval $[0, 3]$.

Look at $f'(x) = 3x^2 - 3 \to f'(x) = 0$ at $x = \pm 1$. Therefore on $[0,3]$, we only need consider $f(0) = 0$, $f(1) = -2$, $f(3) = 18$ to conclude: min occurs at $x=1$, value -2; max at $x=3$, value 18

2. (15 points) (a) Explain what information we get from the signs (positive or negative) and the zero values of both the derivative $f'(x)$ and the second derivative $f''(x)$.

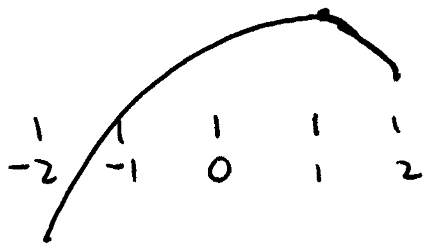
$f' > 0$: f is increasing, $f' < 0$: f is decreasing, $f' = 0$ - can't say

$f'' > 0$: bending up, $f'' < 0$: bending down, $f'' = 0$ - can't say

(b) Give a brief description of **WHY** your answer concerning the sign of $f'(x)$ is correct.

When $f' > 0$ then $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} > 0$ so $\frac{f(x) - f(a)}{x - a}$ is also > 0 for x near a, so $f(x) - f(a)$ has same sign as $x - a$

(c) Sketch a graph of a function f on $[-2, 2]$ satisfying all of the following conditions $f'(x) > 0$ for $-2 \le x < 1$, $f'(1) = 0$, $f'(x) < 0$ for $1 < x \le 2$ with $f''(x) < 0$ for all $-2 \le x \le 2$.

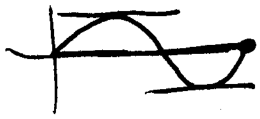


3. (10 points) (a) State the **Mean Value Theorem** with hypotheses.

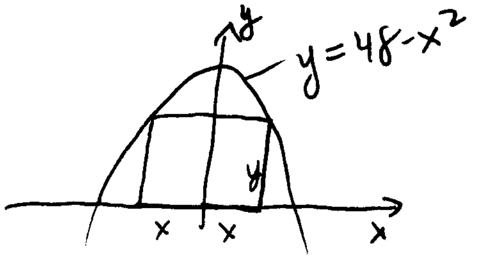
See text

(b) For the function $\sin x$ on the interval $[0, 2\pi]$, find all points c whose existence the Mean Value theorem guarantees.

$$\frac{f(2\pi) - f(0)}{2\pi - 0} = 0 \rightarrow \cos c = 0, \quad c = \pi/2, \frac{3\pi}{2}$$



4. (15 points) A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 48 - x^2$. What is the largest area the rectangle can have and what are its dimensions?



Area $= 2x \cdot y = 2x(48 - x^2)$

largest: when $0 = \frac{d}{dx}[96x - 2x^3] = 96 - 6x^2$

so $x^2 = \frac{96}{6} = 16$

$x = 4$

Dimensions: Base 8

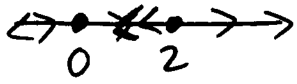
Height $48 - 16 = 32$

Area 256

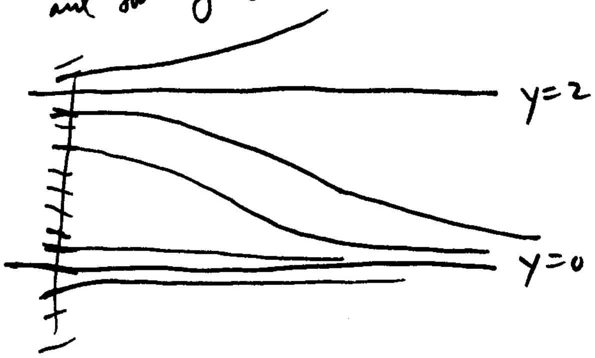
5. (10 points) Draw a phase line and also sketch some solution curves for the autonomous differential equation $\frac{dy}{dx} = y\,(y-2)$. What are the equilibrium values and which, if any, is stable?

$\frac{dy}{dx} = y(y-2)$ Equilibria when $\frac{dy}{dx} = 0 \Leftrightarrow y=0,\ y=2$

$\frac{dy}{dx}$ is positive for $y>2$, negative for $0<y<2$ and for $y<0$ → PHASE LINE



Curves



$y=0$ is STABLE

$y=2$ is UNSTABLE

6. (15 points) Newton's method uses the tangent line approximation to obtain approximate solutions to $f(x) = 0$. Explain how the method leads to the formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Find the next two values if we start the method to solve $x^2 - 6 = 0$ at $x_1 = 3$. You need not simplify answers if they are readable.

Method: Pick x_{n+1} using tangent line thru $(x_n, f(x_n))$: $y - f(x_n) = f'(x_n)\cdot(x - x_n)$ so tangent line hits $(x_{n+1}, 0)$

ALGEBRA: $-f(x_n) = f'(x_n)(x_{n+1} - x_n)$ yields $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

If $x_1 = 3$, $f(x_1) = 3^2 - 6 = 3$, $f'(x_1) = 2x_1 = 6$ so $x_2 = x_1 - 3/6 = 3 - \frac{1}{2} = 2.5$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.5 - \frac{(2.5)^2 - 6}{2(2.5)} = 2.5 - \frac{.25}{5} = 2.45$

Note: $(2.45)^2 = 6.0025$

7. (10 points) For each integration formula below, decide if it is right or wrong and give a brief reason for each answer.

(a) $\int x \sin x \, dx = -x \cos x + C$ (b) $\int \sqrt{2x+1}\, dx = \frac{(2x+1)^{\frac{3}{2}}}{3} + C$

(a) WRONG

$\frac{d}{dx}(-x\cos x + c)$

$= -\cos x + (-x)\cdot(-\sin x)$

$= -\cos x + x\cos x$

using PRODUCT RULE

(b) RIGHT

$\frac{d}{dx}\left[\frac{(2x+1)^{3/2}}{3} + c\right]$

$= \frac{1}{3}\cdot\frac{3}{2}(2x+1)^{1/2}\cdot 2 = (2x+1)^{1/2}$

CHAIN RULE!

8. (15 points) Find all the curves $y = f(x)$ satisfying all the following properties and explain why there are no others:

(a) $\frac{d^2y}{dx^2} = 6x$ (b) The graph goes through the point $(0, 2)$ and its tangent line at that point is horizontal.

From $y'' = 6x$ we get $y' = 3x^2 + A$ ← constant

and then $y = x^3 + Ax + B$ ← second constant

slope at $(0,2)$ is 0 implies $A = 0$

$y = 2$ at $x = 0$ then gives $B = 2$

Answer $y = x^3 + 2$

There are no others because two antiderivatives lead to two integration constants (A, B) which were specified by knowing y and y' at $x = 0$