Math 113 – Exam 2 – Fall 2004 – Professor Sachs

Name: SOLUTION

Answer each question on this paper. Read them carefully and answer clearly. Exam ends at 2:20pm. The GMU Honor Code is in effect. The exam is worth 100 points.

1. (10 points) Find an equation for the tangent line to the curve $y = 1/x^3$ at the point $(2, 1/8)$.

$y = x^{-3} \Rightarrow \frac{dy}{dx} = -3x^{-4}$ so slope at $x=2$ is $-3/16$

Pt-slope: $y - 1/8 = -3/16\,(x-2)$

2. (15 points) Find derivatives of the functions below at the points indicated. You must show your work.

(*a*) $\sqrt{4-x^2}$ at $x = 1$.

$y = \sqrt{4-x^2} = \sqrt{u}$, $u = 4-x^2$

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2}u^{-1/2}\cdot(-2x)$: at $x=1$, $u = 4-1=3$

so derivative is $\frac{-1}{\sqrt{3}}$

(*b*) $\sin(2\pi(t-1))$ at $t = 1/2$.

$y = \sin(u)$, $u = 2\pi(t-1) \Rightarrow$ deriv. is $\cos(u) \cdot 2\pi$: at $t = 1/2$, $u = 2\pi(-\frac{1}{2}) = -\pi$

($\cos(u)$ is $\frac{dy}{du}$, 2π is $\frac{du}{dt}$)

ANS $2\pi\cos(-\pi) = -2\pi$

(*c*) $(1 + x^{-1/2})^{-1/2}$ at $x = 1/9$.

$y = u^{-1/2}$, $u = 1+x^{-1/2}$; at $x = 1/9$, $u = 1+(1/9)^{-1/2} = 1+3 = 4$

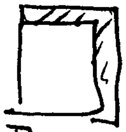
$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{2}u^{-3/2}\cdot(-1/2)x^{-3/2} = 1/4\,(4)^{-3/2}\,(1/9)^{-3/2} = 1/4 \cdot \frac{1}{8} \cdot 27$

$= \frac{27}{32}$

3. **(10 points)** **State the product rule and explain why the product rule ought to be true.**

If f and g are differentiable at c, so is $f(x)\cdot g(x)$ and derivative at c is $f'(c)\cdot g(c) + f(c)\,g'(c)$.

Why? Either algebra (using definition) or picture



– change in product

4. **(15 points)** **Suppose that the functions $f(x)$ and $g(x)$ and their derivatives have the values at $x = 0, x = 1$ and $x = 3$:**

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	3	2	−1
1	3	0	4	−2
3	6	−5	5	−4

Find the derivatives at the indicated points:

(a) $f(g(x))$ **at** $x = 1$

$f'(g(1))\cdot g'(1) = f'(0)\cdot g'(1) = 2\cdot(-2) = \boxed{-4}$

(b) $\sqrt{g(x)}$ **at** $x = 0$

$\frac{1}{2}\, g(x)^{-1/2}\cdot g'(x)$ at $x=0$ $= \frac{1}{2}\cdot\frac{1}{\sqrt{3}}\cdot(-1) = \boxed{-\frac{1}{2\sqrt{3}}}$

(c) The inverse function of f, $f^{-1}(x)$, **at** $x = 3$

Deriv. of f^{-1} at $3 = \frac{1}{f'(1)}$ since $f(1) = 3 \Leftrightarrow f^{-1}(3) = 1$

$= \boxed{1/4}$

5. (10 points) State what the derivatives of $\cos\theta$ and $\sin\theta$ are and explain why these are correct, using **either** their definition in terms of radian measure and the unit circle **or** using trig identities and the definition of derivative as limit.

Derivative formulas! $\frac{d}{d\theta}\cos\theta = -\sin\theta$, $\frac{d}{d\theta}\sin\theta = \cos\theta$

Why? Either trig identities, definition of derivatives as limits OR Picture of unit circle, change in x, y coordinates go in tangent direction → rotated by $\pi/2$ ahead of θ itself for direction.

6. (15 points) Find the **differentials** dy of each function:

(a) $y = x\ln(x)$

$$dy = \frac{d}{dx}\left(x\ln(x)\right)\cdot dx = \left[1\cdot\ln x + x\cdot\frac{1}{x}\right]dx$$

↑ PRODUCT RULE

so $\boxed{dy = (\ln x + 1)\,dx}$

(b) $y = \frac{x}{1+x^2}$

$$dy = \frac{d}{dx}\left(\frac{x}{1+x^2}\right)dx = \left(\frac{1\cdot(1+x^2) - x\cdot 2x}{(1+x^2)^2}\right)dx$$

↑ QUOTIENT

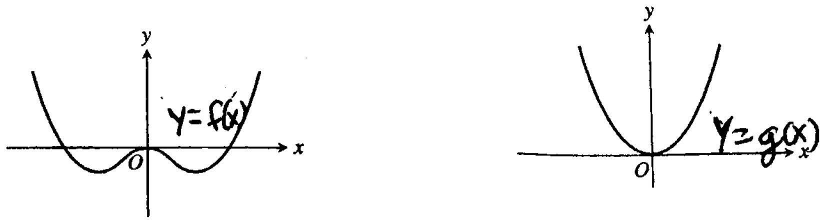
so $\boxed{dy = \frac{1-x^2}{(1+x^2)^2}\,dx}$

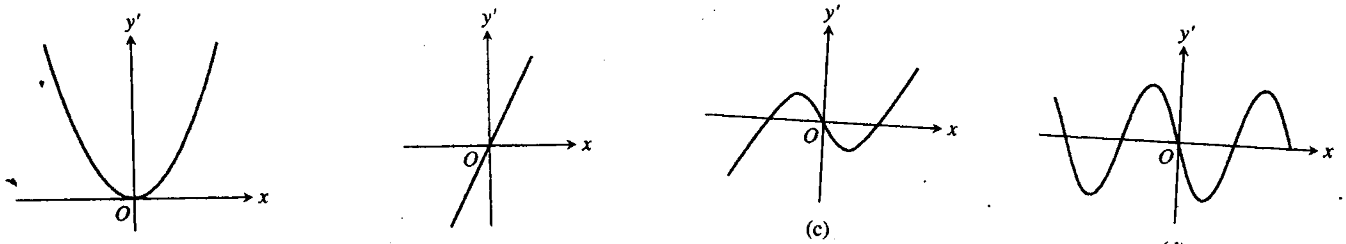
(c) $y = e^{-3x}$

$\boxed{dy = -3e^{-3x}\,dx}$

chain rule

7. (10 points) For the functions $f(x)$ and $g(x)$ graphed below, decide which of the derivative graphs is $f'(x)$ and which is $g'(x)$ and explain your choices.





(a) (b) (c) (d)

(b) is $g'(x)$, (c) is $f'(x)$

Why? (b) g increases for $x>0$, decreases for $x<0$, more steeply as $|x|$ increases

(c) f decreases up to some minimum for a value below 0, increases until $x=0$, then decreases to another local min, then increases.

Note: (a) would involve increasing function for all $x \neq 0$

(d) would involve 3 increasing phases and 2 decreasing.

8. (15 points) For the function $y = e^{-x^2}$, find its first derivative and second derivative, and then find all values of x where $y' = 0$ and all values of x where $y'' = 0$.

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^{-x^2} \cdot (-2x) = -2x\, e^{-x^2}$$

$$y = e^u \qquad u = -x^2$$

$$\frac{dy}{du} = e^u \qquad \frac{du}{dx} = -2x$$

$$\frac{dy}{dx} = 0 \iff x = 0 \text{ since } e^u \neq 0.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-2x\, e^{-x^2}\right) = \underbrace{-2\, e^{-x^2} + (-2x)(e^{-x^2})'}_{\text{product rule}}$$

$$= -2\, e^{-x^2} + (-2x)(-2x)\, e^{-x^2}$$

$$= (-2 + 4x^2)\, e^{-x^2}$$

$$\text{so } y'' = 0 \iff 4x^2 - 2 = 0 \iff x^2 = \tfrac{1}{2} \iff x = \pm\frac{1}{\sqrt{2}}$$