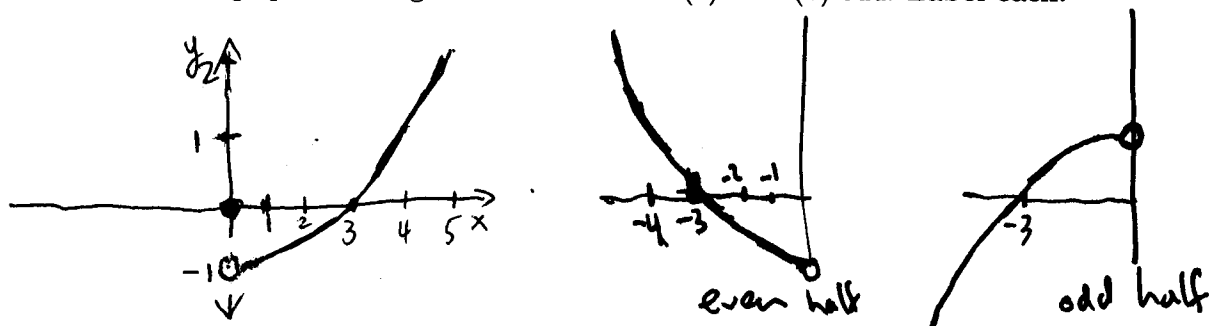


Name: SOLUTIONS

Answer each question on this paper. Read them carefully and answer clearly. Exam ends at 2:20pm. The GMU Honor Code is in effect. The exam is worth 100 points.

1. (10 points) A portion of the graph of a function defined on the interval  $[-5, 5]$  is shown. Complete the graph assuming that the function is (a) even (b) odd. Label each.



2. (15 points) (a) If  $f$  is a one-to-one function and  $f(x)$  is never zero, show that  $g(x) = 1/f(x)$  is also one-to-one.

If  $g(a) = g(b)$  then  $\frac{1}{f(a)} = \frac{1}{f(b)}$  which implies  $f(a) = f(b)$ .  
 But  $f$  is one-to-one, so  $a = b$ . This shows  $g$  is one-to-one.

- (b) Assuming (a) is true even if you had trouble showing it fully, this shows that  $g$  has an inverse function. When the graph of  $f(x)$  only for  $1 \leq x \leq 5$  is the line through the points  $(1, 8)$  and  $(5, 20)$ , find a formula for the inverse function of  $g(x)$  and describe its domain, given the restriction  $1 \leq x \leq 5$  for the domain of  $f$ .

$$f(x) = 8 + m \cdot (x-1) \text{ [pt-slope]} \text{ and } m = \frac{20-8}{5-1} = \frac{12}{4} = 3 \text{ so } f(x) = 3(x-1) + 8$$

$$g(x) = \frac{1}{f(x)} = \frac{1}{3x+5} \text{ and for } 1 \leq x \leq 5, \frac{1}{20} \leq g(x) \leq \frac{1}{8} = 3x+5$$

$g^{-1}$  maps  $[\frac{1}{20}, \frac{1}{8}]$  onto  $[1, 5]$  and to find its formula, solve

$$y = g(x) = \frac{1}{3x+5} \text{ for } x \text{ in terms of } y: \frac{1}{y} = 3x+5, 3x = \frac{1}{y} - 5$$

$$\boxed{x = \frac{1}{3} \left( \frac{1}{y} - 5 \right)}$$

3. (10 points) If the function  $y = (1/3) \cdot 2^x$  is written as an exponential in base  $e$  (meaning in the form  $y_0 e^{kx}$ ), what are the numbers  $y_0$  and  $k$ ?

$y_0 e^{kx} = \frac{1}{3} \cdot 2^x$  forces  $y_0 = \frac{1}{3}$  and  $e^{kx} = 2^x$   
 [for example, use  $x=0$  or take  $\ln$  of both sides]. Then  $e^k$  must equal 2 so  $k = \ln(2)$ . Check  $\frac{1}{3} e^{(\ln 2)x} = \frac{1}{3} \cdot 2^x$  ✓

Alternate solution: take  $\ln$  of both:

$$\ln(y_0 e^{kx}) = \ln(y_0) + \ln(e^{kx}) = \ln(y_0) + kx$$

$$\ln\left(\frac{1}{3} \cdot 2^x\right) = \ln\left(\frac{1}{3}\right) + \ln(2^x) = \ln\left(\frac{1}{3}\right) + (\ln 2) \cdot x$$

4. (15 points) A ball of pizza dough dropped from the Leaning Tower of Pisa falls a distance  $y = 16t^2$  feet after  $t$  seconds.

Describe algebraically and graphically the average speed of the pizza dough over the following intervals: from 2 seconds to 3 seconds; from 2 seconds to  $t$  seconds. Then find the instantaneous speed at 2 seconds. **YOU MUST SHOW SOME WORK FOR INSTANTANEOUS SPEED.**

$$y(2) = 16 \cdot 4 = 64 \text{ ft}, \quad y(3) = 16 \cdot 9 = 144 \text{ ft} \text{ so}$$

$$\text{avg speed, 2 sec to 3 sec} = \frac{144 - 64}{3 - 2} = 80 \text{ ft/sec}$$

$$\text{avg speed, 2 sec to } t \text{ sec} = \frac{16t^2 - 64}{t - 2} = 16t + 32 = 16(t + 2)$$

all ft/sec

OK      better

$$\text{limit of avg speed, 2 sec to } t \text{ sec, as } t \rightarrow 2 \text{ sec}$$

$$= \text{instantaneous speed} = \lim_{t \rightarrow 2} 16t + 32 = 64 \text{ ft/sec}$$

5. (10 points) Describe how the radian measure of an angle is defined. Then describe the trigonometric functions  $\cos \theta$  and  $\sin \theta$  in terms of the radian measure and the unit circle.

Radian measure = distance around unit circle  
 [fraction of circumference ( $2\pi$ )]  
 $(\cos \theta, \sin \theta)$  represent  $(x, y)$  coordinates on circle.

6. (15 points) Find the limits:

$$(a) \lim_{x \rightarrow 2} \frac{x^2 + 4}{x + 4} = \frac{\lim_{x \rightarrow 2} x^2 + 4}{\lim_{x \rightarrow 2} x + 4} = \frac{4 + 4}{2 + 4} = \frac{8}{6} \stackrel{\text{either}}{\text{OK}} \rightarrow \frac{4}{3}$$

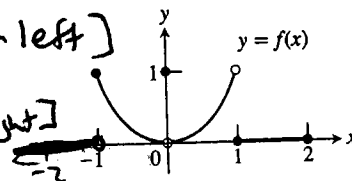
$$(b) \lim_{y \rightarrow 1} \sqrt{4y^2 + 4y + 1} = \sqrt{\lim_{y \rightarrow 1} 4y^2 + 4y + 1} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$(c) \lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} \text{ CAN'T DO } \frac{0}{0} \text{ so } \underline{\text{DIVIDE}} \text{ THEN CONQUER}$$

$$\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = \lim_{x \rightarrow -3} x - 3 = -3 - 3 = -6$$

7. (10 points) For the function  $f(t)$  graphed below, find the following limits or explain why they do not exist:

- (a)  $\lim_{t \rightarrow 0^-} f(t) = 0$  [from graph near 0 on left]
- (b)  $\lim_{t \rightarrow 0^+} f(t) = 0$  [from graph near 0 on right]
- (c)  $\lim_{t \rightarrow -1} f(t)$ : DOES NOT EXIST - Left limit is 0, Right limit is 1



8. (15 points) (a) Give a formal definition of limit of a function  $f$ . Limit of  $f(x)$  is  $L$  as  $x \rightarrow x_0$ :

Given any  $\epsilon > 0$ , there is a  $\delta$  such that for all  $0 < |x - x_0| < \delta$ ,  $|f(x) - L| < \epsilon$

(b) For the function  $f(x) = 1/x$ , find a  $\delta > 0$  such that for all  $x$  satisfying  $0 < |x - 1| < \delta$ , the inequality  $|f(x) - 1| < \frac{1}{2}$  holds.

$$\left| \frac{1}{x} - 1 \right| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < \frac{1}{x} - 1 < \frac{1}{2} \Leftrightarrow \frac{1}{2} < \frac{1}{x} < \frac{3}{2} \Leftrightarrow 2 > x > \frac{2}{3}$$

This forces  $\delta \leq \frac{1}{3}$  [shorter distance from 1 to edges]

(c) Explain part (b) graphically.

