MATH 325, Section 001
Spring, 2011
Exam 2 Name Solutions

1. At a roadblock, the driver's licenses of 10 drivers are collected. How many ways are there of returning the licenses so that none of the drivers get their own licenses back?

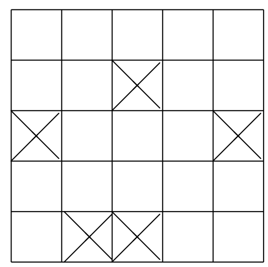
The problem asks for the number of derangements of 10 things. That number is

$$10!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{1}{10!}\right).$$

2. At a roadblock, the driver's licenses of 10 drivers are collected. How many ways are there of returning the licenses so that exactly 4 of the drivers get their own licenses back?

There are $\binom{10}{4}$ ways of choosing the 4 drivers who will receive their own licenses. For each of these ways, the number of ways of returning the remaining 6 licenses to the remaining 6 drivers so that none of them get their own license is the number of derangements of 6 things, or $6!(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!})$. Therefore, the number of ways of returning the licenses so that exactly 4 get their own licenses is $\binom{10}{4}6!(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!})$.

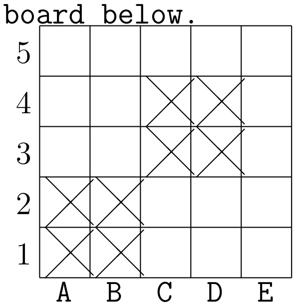
3. How many ways are there of placing 5 non-attacking rooks on the chess board below, where rooks cannot be placed in the forbidden positions marked by x's?



Let B_1 be the block consisting of the two forbidden squares in the third row from the top, and let B_2 be block consisting of the three remaining forbidden squares. Then B_1 and B_2 form independent blocks. Let r_k denote the number of ways of placing k non-attacking rooks in forbidden positions. Then $r_1 = 5$. To find r_2, we note that there is 1 way of placing two non-attacking rooks in B_2 and there are $2 \cdot 3 = 6$ ways of placing one rook in each of B_1 and B_2, so $r_2 = 1 + 6 = 7$. In order to place 3 non-attacking rooks in forbidden positions, we must place 2 in B_2 and 1 in B_1. There are $1 \cdot 2 = 2$ ways of doing this so $r_3 = 2$. Since the forbidden positions all lie in 3 rows, $r_4 = r_5 = 0$. Therefore, the number of ways of placing 5 non-attacking rooks in a way that avoids forbidden positions is $5! - r_1 4! + r_2 3! - r_3 2! + r_4 1! - r_5 = 5! - 5(4!) + 7(3!) - 2(2!) = 38$.

4. How many ways are there of assigning office cubicles, labeled 1, 2, 3, 4, and 5, to five new employees, A, B, C, D, and E, if neither A nor B can be placed in cubicles 1 or 2, and neither C nor D can be placed in cubicles 3 or 4?

We need to find the number of ways of placing five non-attacking rooks in the board below.



If we define B_1 and B_2 to be the forbidden 2×2 block in the lower left corner and B_2 to be the block consisting of the remaining forbidden positions, then B_1 and B_2 are independent blocks. Letting r_k be the number of ways of placing k non-attacking rooks in forbidden positions, we see that $r_1 = 8$, $r_2 = (4)(4)+(2)(2) = 20$, $r_3 = (2)(4)(2) = 16$, $r_4 = (2)(2) = 4$, and $r_5 = 0$. Therefore, the number of arrangements is $5! - 8(4!) + 20(3!) - 16(2!) + 4(1!) = 20$.

5. Let a_n denote the number of n-bit strings of 0s and 1's that contain the string 110.

   (a) Find a recurrence relation satisfied by (a_n) and give enough initial conditions to allow you to find a_n for any n.

   Any string of length $n-1$ which contains the string 110 can be followed by either a 0 or a 1 to give a string of length n that contains the string 110, so there are $2a_{n-1}$ n-bit strings that start with an $n-1$ bit string containing 110. It is also possible to get an n-bit string containing 110 by starting with an $n-3$ bit string that does not contain 110 and putting a 110 at the end of it. There are 2^{n-3}

$n-3$-bit strings, of which a_{n-3} contain the string 110, so there are $2^{n-3} - a_{n-3}$ n-bit strings consisting of an $n-3$-bit string not containing 110 followed by 110. Therefore,

$$a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}.$$

Since this is an order three recurrence relation, we need three initial conditions. We get

$$a_1 = a_2 = 0, a_3 = 1.$$

(b) Find a_6.

$$a_4 = 2a_3 + 2^{4-3} - a_1 = 2 + 2 - 0 = 4.$$
$$a_5 = 2a_4 + 2^{5-3} - a_2 = 8 + 4 - 0 = 12.$$

Therefore,

$$a_6 = 2a_5 + 2^{6-3} - a_3 = 24 + 8 - 1 = 31.$$

6. Suppose that (a_n) is a recurrence relation satisfying $a_n = a_{n-1} + 8(n-1)$ with the initial condition $a_1 = 1$. Use mathematical induction to show that $a_n = (2n-1)^2$ for $n = 1, 2, 3, \cdots$.

Proof:
Base step: $a_1 = 1 = 1^2 = (2 \cdot 1 - 1)^2$, so $a_n = (2n-1)^2$ for $n = 1$.

Inductive step: Suppose that $n > 1$ and assume that $a_k = (2k-1)^2$ for $k < n$. Then $a_n = a_{n-1} + 8(n-1) = (2(n-1)-1)^2 + 8(n-1) = (2n-3)^2 + 8n - 8 = 4n^2 - 12n + 9 + 8n - 8 = 4n^2 - 4n + 1 = (2n-1)^2$. The second equality holds by the induction hypothesis since $n - 1 < n$.

7. Find a solution of the recurrence relation (a_n) satisfying $a_n = 2a_{n-1} + 3a_{n-2}$ with the initial conditions $a_0 = 1$, $a_1 = 7$.

   The characteristic polynomial is $p(x) = x^2 - 2x - 3 = (x-3)(x+1)$. Therefore, the general solution of the recurrence relation is $a_n = C_1 3^n + C_2(-1)^n$. Since $a_0 = 1$, $C_1 + C_2 = 1$. Since $a_1 = 7$, $3C_1 - C_2 = 7$. Solving for C_1 and C_2 gives $C_1 = 2$, $C_2 = 1$, so we get the solution $a_n = 2(3^n) + (-1)^n$.

8. Find the general solution of the recurrence relation $a_n = 2a_{n-1} + 8a_{n-2} + 3^n$.

   The characteristic polynomial of the homogeneous recurrence relation $a_n = 2a_{n-1} + 8a_{n-2}$ is $p(x) = x^2 - 2x - 8 = (x-4)(x+2)$, so the general solution of the homogeneous recurrence relation is $a_n = C_1 4^n + C_2(-2)^n$. We look for a solution of the given recurrence relation of the form $a_n = K3^n$. Then $a_{n-1} = K3^{n-1}$ and $a_{n-2} = K3^{n-2}$, so K must satisfy

$$K3^n = 2K3^{n-1} + 8K3^{n-2} + 3^n.$$

   Dividing by 3^{n-2} gives

$$K3^2 = 2K(3) + 8K + 3^2,$$

   or

$$9K = 6K + 8K + 9.$$

   Solving for K gives $K = \frac{-9}{5}$ so a solution is $a_n = (\frac{-9}{5})3^n = -\frac{3^{n+2}}{5}$. Adding this to the general solution of the homogeneous recurrence relation gives the general solution

$$a_n = C_1 4^n + C_2(-2)^n - \frac{3^{n+2}}{5}.$$

9. (10 points) Find a recurrence relation for the sequence (a_n) if (a_n) and (b_n) satisfy

$$\begin{aligned} a_n &= a_{n-1} + b_{n-1}. \\ b_n &= a_{n-1} - b_{n-1}. \end{aligned}$$

Your solution should not involve any term b_k.

From the first equation, we get $b_{n-1} = a_n - a_{n-1}$, so, replacing $n-1$ with n we get $b_n = a_{n+1} - a_n$. Substituting in the second equation gives $a_{n+1} - a_n = a_{n-1} - (a_n - a_{n-1})$, or $a_{n+1} = 2a_{n-1}$. Replacing $n+1$ with n gives

$$a_n = 2a_{n-2}.$$