MATH 290, Section 001 — Name: Solutions

Fall, 2011

Exam 1 — Student ID number: ____________

1. Complete the following truth table for the sentence $(\sim p) \Rightarrow (p \vee q)$.

| $p$ | $q$ | $\sim p$ | $p \vee q$ | $(\sim p) \Rightarrow (p \vee q)$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | T | T |
| F | T | T | T | T |
| F | F | T | F | F |

2. Write the converse and the contrapositive of the statement "If $X$ is compact, then $X$ is normal."

Converse:

If $X$ is normal, then $X$ is compact.

Contrapositive:

If $X$ is not normal, then $X$ is not compact.

3. State the negation of each of the following statements in a non-trivial way. (In other words, do not simply put a phrase like "It is not the case that" in front of the statement.)

   (a) If the contestant is able to eat one gallon of mayonnaise in five minutes, then he is the winner of the gold medal in mayonnaise eating.

   Negation: The contestant is able to eat one gallon of mayonnaise in five minutes, but he is not the winner of the gold medal in mayonnaise eating.

   (b) The subgroup $H$ is normal and the ring $R$ is simple.

   Negation: Either the subgroup $H$ is not normal or the ring $R$ is not simple.

4. Write the following statement using quantifiers and connectives. Assume that the universe is the set of real numbers. "For any real numbers $x$ and $y$ such that $x < y$, there is a rational number $q$ such that $x < q < y$ and $y < q + 1$."

Answer:

$(\forall x)(\forall y)(x < y \Rightarrow (\exists q)(q \texttt{ is rational} \wedge (x < q < y) \wedge (y < q + 1)))$

5. Write a useful negation of the statement $(\exists x)(x > 0 \wedge (\forall y)(y > 0 \Rightarrow x < y))$. Do not simply put a negation symbol in front of the given statement.

The negation of the statement $(\exists x)P(x)$ is $(\forall x) \sim P(x)$. The negation of $p \wedge q$ is $\sim p \vee \sim q$. The negation of $(\forall x)P(x)$ is $(\exists x) \sim P(x)$. The negation of $p \Rightarrow q$ is $p \wedge \sim q$. Therefore, the negation of the given statement is

$$(\forall x)(x \leq 0 \vee (\exists y)(y > 0 \wedge x \geq y).$$

Answer:

$(\forall x)(x \leq 0 \vee (\exists y)(y > 0 \wedge x \geq y)$

6. Let the universe consist of all integers. Prove that if $3|a$ and $9|b$, then $9|(a^2 + b)$.

Proof:

Suppose that $3|a$ and $9|b$. Then there are integers $m$ and $n$ such that $a = 3m$ and $b = 9n$. Therefore, $a^2 + b = (3m)^2 + (9n) = 9m^2 + 9n = 9(m^2 + n)$. Letting $k = m^2 + n$ gives $a^2 + b = 9k$, so $9|a^2 + b$. ■

7. Let the universe consist of all real numbers. Prove that if $|x| > 3$, then $|x-1| > 2$.

Proof:

Suppose $|x| > 3$. We give a proof by cases.
Case (1). If $x > 0$, then $|x| = x$. Since $|x| > 3$, $x > 3$, so $x - 1 > 2 > 0$. Therefore, $|x-1| = x - 1 > 2$.
Case (2). If $x < 0$, then $|x| = -x > 3$, so $x < -3$. Therefore, $x - 1 < -4$, so $|x-1| > 4 > 2$. ■

8. Let the universe consist of the set $\mathbb{R}$ of real numbers. Prove $(\forall x)((x \text{ rational} \wedge x \neq 0) \Rightarrow x\sqrt{2} \text{ is irrational})$. You may use what you know about $\sqrt{2}$ and about basic properties of rational numbers, but you may not simply state that the product of a nonzero rational number with an irrational number is irrational.

Proof:

Let $x$ be a rational number such that $x \neq 0$. Suppose that $x\sqrt{2}$ were rational. Since $x \neq 0$, we can divide $x\sqrt{2}$ by $x$ to get $\sqrt{2}$. But the quotient of rational numbers is rational, so this would mean that $\sqrt{2}$ is rational. However, we know that $\sqrt{2}$ is irrational, so we have a contradiction. Therefore, our assumption that $x\sqrt{2}$ is rational is incorrect. ■