MATH 214, Section 001 Name Solutions

Spring, 2011

Exam 4 Student ID number____________

The following table might be helpful for some of the problems below.

Some Laplace Transforms		
	$f(t)$	$\mathcal{L}(f(t))(s)$
1.	1	$\frac{1}{s}$
2.	t	$\frac{1}{s^2}$
3.	e^{at}	$\frac{1}{s-a}$
4.	$\sin(at)$	$\frac{a}{s^2+a^2}$
5.	$\cos(at)$	$\frac{s}{s^2+a^2}$
6.	$f^{(n)}(t)$	$s^n\mathcal{L}(f(t))(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
7.	$u_c(t)f(t-c)$	$e^{-cs}\mathcal{L}(f(t))(s)$
8.	$e^{ct}f(t)$	$\mathcal{L}(f(t))(s-c)$

1. Write the function $f(t) = \begin{cases} 3 & \text{If } t < 2 \\ t^2 - 1 & \text{If } 2 \le t < 3 \\ 8 & \text{If } 3 \le t \end{cases}$ as a sum of functions of the form $h(t)u_c(t)$, where $u_c(t) = \begin{cases} 0 & \text{If } t < c. \\ 1 & \text{If } t \ge c. \end{cases}$

Since the characteristic function $\chi_{[c,d)}(t) = \begin{cases} 0 & \text{If } t < c, \\ 1 & \text{If } c \le t < d, \\ 0 & \text{If } d \le t, \end{cases}$ of the interval $[c,d)$ can be written as $\chi_{[c,d)}(t) = u_c(t) - u_d(t)$, $f(t) = 3(u_0(t) - u_2(t)) + (t^2 - 1)(u_2(t) - u_3(t)) + 8u_3(t) = 3u_0(t) + (t^2 + 4)u_2(t) + (9 - t^2)u_3(t)$. [Note: If you are thinking of f as being defined for negative values of t as well as positive, you can add $3(1 - u_0(t))$ to the solution above.]

Answer:
$$f(t) = 3u_0(t) + (t^2 + 4)u_2(t) + (9 - t^2)u_3(t)$$

2. Let $h(t) = \begin{cases} 0 & \text{If } t < 2\pi. \\ \sin t & \text{If } t \ge 2\pi. \end{cases}$ Find the Laplace transform $\mathcal{L}(x(t))(s)$ of a solution of the initial value problem $\frac{dx}{dt} - 2x = h(t)$, $x(0) = 5$.

$h(t) = u_{2\pi}(t)\sin t$. Since $\sin t = \sin(t - 2\pi)$, $h(t) = u_{2\pi(t)}\sin(t - 2\pi)$. Therefore, $\mathcal{L}(h(t)) = e^{-2\pi s}\mathcal{L}(\sin t)(s) = \frac{e^{-2\pi s}}{s^2+1}$. Taking Laplace transforms of both sides of the equation gives $s\mathcal{L}(x(t))(s) - x(0) - 2\mathcal{L}(x(t))(s) = \frac{e^{-2\pi s}}{s^2+1}$, or $s\mathcal{L}(x(t))(s) - 5 - 2\mathcal{L}(x(t))(s) = \frac{e^{-2\pi s}}{s^2+1}$. Solving gives $\mathcal{L}(x(t))(s) = \frac{5}{s-2} + \frac{e^{-2\pi s}}{(s-2)(s^2+1)}$.

$$\mathcal{L}(x(t))(s) = \frac{5}{s-2} + \frac{e^{-2\pi s}}{(s-2)(s^2+1)}$$

3. Find the inverse Laplace transform of the function $F(s) = \frac{s}{s^2-6s+34}$.

`Completing the square in the denominator gives` $F(s) = \frac{s}{(s-3)^2+25} = \frac{s-3}{(s-3)^2+25} + (\frac{3}{5})\frac{5}{(s-3)^2+25}$. `Since` $\mathcal{L}^{-1}(\frac{s}{s^2+25})(t) = \cos(5t)$, `and` $\mathcal{L}^{-1}(\frac{5}{s^2+25})(t) = \sin(5t)$, `the last line of the table gives` $\mathcal{L}^{-1}(F(s))(t) = e^{3t}(\cos(5t) + (\frac{3}{5})\sin(5t))$.

$\mathcal{L}^{-1}(F(s))(t) =$ $e^{3t}(\cos(5t) + (\frac{3}{5})\sin(5t))$

4. Solve the initial value problem $\frac{dx}{dt} - x = \begin{cases} 0 & \text{If } t < \pi \text{ ,} \\ \sin t & \text{If } t \geq \pi \text{ ,} \end{cases}$, $x(0) = 0$.
[Hint: It might help to know that the partial fraction expansion for $\frac{2}{(s-1)(s^2+1)}$ is $\frac{1}{s-1} - \frac{s+1}{s^2+1}$.]

`Taking Laplace transforms of both sides of the ODE and using the initial condition gives` $s\mathcal{L}(x)(s) - 0 - \mathcal{L}(x)(s) = -e^{-\pi s}\mathcal{L}(\sin(t-\pi)(s) = -\frac{e^{-\pi s}}{s^2+1}$, `so` $\mathcal{L}(x)(s) = \frac{-e^{-\pi s}}{(s-1)(s^2+1)}$. `Therefore, from the hint`

$$\mathcal{L}(x)(s) = -(\frac{1}{2})(\frac{e^{-\pi s}}{s-1} - \frac{se^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+1}).$$

`Therefore,`

$$x(t) = -(\frac{1}{2})(u_\pi(t))(e^{t-\pi} - \cos(t-\pi) - \sin(t-\pi)) = -(\frac{1}{2})(u_\pi(t))(e^{t-\pi} + \cos t + \sin t).$$

$x(t) =$ $-(\frac{1}{2})(u_\pi(t))(e^{t-\pi} + \cos t + \sin t)$

5. Use the fact that $\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 & \frac{1}{2} \\ 1 & -1 & \frac{1}{2} \\ -1 & 2 & -\frac{1}{2} \end{pmatrix}$ to solve the system of equations $\begin{cases} -x + y & = & 4 \\ y + z & = & -2 \\ 2x + 2y + 2z & = & 8 \end{cases}$

The system can be written as $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix}$, where $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$. Therefore, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 & -1 & \frac{1}{2} \\ 1 & -1 & \frac{1}{2} \\ -1 & 2 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ -12 \end{pmatrix}$, so $x = 6$, $y = 10$, $z = -12$.

$$x = 6$$
$$y = 10$$
$$z = -12$$

6. Find all eigenvalues of the matrix $A = \begin{pmatrix} 7 & -4 \\ 8 & -5 \end{pmatrix}$

The eigenvalues are the roots of the polynomial $p(\lambda) = det(A-\lambda I) = \det\begin{pmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{pmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$. Therefore, the eigenvalues are $\lambda = -1$ and $\lambda = 3$.

Answer:
$\lambda = -1$ and $\lambda = 3$

7. Let $A = \begin{pmatrix} -1 & -6 \\ 2 & 6 \end{pmatrix}$. Then eigenvalues for A are $\lambda = 2$ and $\lambda = 3$ with corresponding eigenvectors $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, respectively. Find $exp(A)$.

If P is the matrix whose columns are the given eigenvectors, that is, $P = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, then $P^{-1}AP$ is the diagonal matrix

$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. `Therefore,` $A = PDP^{-1}$ `so` $exp(A) = exp(PDP^{-1}) =$ $P(exp(D))P^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4e^2 - 3e^3 & 6e^2 - 6e^3 \\ -2e^2 + 2e^3 & -3e^2 + 4e^3 \end{pmatrix}$.

$$\boxed{exp(A) = \begin{pmatrix} 4e^2 - 3e^3 & 6e^2 - 6e^3 \\ -2e^2 + 2e^3 & -3e^2 + 4e^3 \end{pmatrix}}$$