MATH 203, Section 001 Fall, 2011 Exam 3

1. Let
$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ -1 & -1 \end{pmatrix}$$
 and let $\vec{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$. Determine whether or not

b is in the column space of A. Be sure that your work makes clear how you are arriving and your answer.

We row reduce the augmented matrix
$$\begin{pmatrix} 1 & 3 & | & 1 \\ 2 & 4 & | & -2 \\ -1 & -1 & | & 3 \end{pmatrix}$$
, to get
 $\begin{pmatrix} 1 & 3 & | & 1 \\ 2 & 4 & | & -2 \\ -1 & -1 & | & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & | & 1 \\ 0 & -2 & | & -4 \\ 0 & 2 & | & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & | & 1 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & | & -5 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{pmatrix}$.
Therefore, $\vec{b} = -5 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$, so \vec{b} is in the column space of A .

 $\boxtimes \vec{b}$ is in the column space of A. $\Box \vec{b}$ is not in the column space of A

2. Let $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 0 & -3 \end{pmatrix}$ and let $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Determine whether or not \vec{b} is in the null space of A. Be sure that your work makes clear how you are arriving and your answer.

$$A\vec{b} = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \neq \vec{0}.$$
 Therefore, \vec{b} is not in the null spaced of A

 \Box \vec{b} is in the null space of A. \boxtimes \vec{b} is not in the null space of A

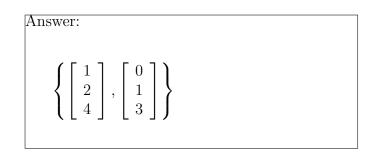
For problems 3 and 4, let
$$A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 2 & 6 & 1 & 0 & 1 \\ 4 & 12 & 3 & -4 & 5 \end{bmatrix}$$
. Then the row-
reduced echelon form of A is $\begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

3. Find a basis for the null space of A.

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ is in the null space of A, then from the rowreduced echelon matrix, we get $x_1 = -3x_2 - 2x_4 + x_5$ and $x_3 =$ $4x_4 - 3x_5 \text{ so } \vec{x} = \begin{bmatrix} -3x_2 - 2x_4 + x_5 \\ x_2 \\ 4x_4 - 3x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} +$ $x_5 \begin{bmatrix} 1\\0\\-3\\0\\1 \end{bmatrix}$. Therefore, a basis is given by $\left\{ \begin{array}{c|c|c} -3 & -2 & -2 \\ 1 & 0 & -3 \\ 0 & -3 & -3 \\ 0 & -3 & 0 \\ 0 & -3 & 0 \\ 0 & -3 & 0 \\ 0 & -3 & 0 \\ 0 & -3 & 0 \\ 1 & 0 \\ 1 \end{array} \right\}.$ Answer:

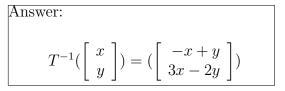
4. Find a basis for the column space of A.

Since the pivot columns of A are the first and third, a basis for the column space is formed by the first and third columns of A. Therefore, a basis for the column space is $\left\{ \begin{bmatrix} 1\\2\\4 \end{bmatrix}, \begin{bmatrix} 0\\1\\3 \end{bmatrix} \right\}$.



5. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x+y \\ 3x+y \end{bmatrix}$. Determine whether or not T is invertible, and if it is, find T^{-1} .

The standard matrix for T is $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$. Since $(2)(1) - (3)(1) = -1 \neq 0$, A is invertible and $T^{-1} = T_{A^{-1}}$. $A^{-1} = \frac{1}{(2)(1) - (3)(1)} \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$. Therefore, $T^{-1}(\begin{bmatrix} x \\ y \end{bmatrix}) = (\begin{bmatrix} -x + y \\ 3x - 2y \end{bmatrix})$.



6. Let
$$A = \begin{bmatrix} 2 & 3 & -1 \\ -2 & 1 & 5 \\ 5 & 2 & 0 \end{bmatrix}$$
. Find $det A$.
Expanding along the third row we get $det A = (5)det \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} - (2)det \begin{bmatrix} 2 & -1 \\ -2 & 5 \end{bmatrix} + (0)det \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix} = (5)(16) - (2)(8) + 0 = 64$.
Answer:
64

7. Suppose that A is a 4×4 matrix such that det A = 3, and that B is obtained from A by interchanging the first and second rows. Find det(2B).

Since B is obtained from A by interchanging two rows, detB = -detA = -3. Since the matrix 2B is obtained from B by multiplying each of four rows by 2, $det(2B) = 2^4 detB = 16(-3) = -48$.

$$det(2B) = -48$$

8. Let
$$A = \begin{bmatrix} 3 & 8 & -2 & 12 \\ 5 & 2 & 0 & -2 \\ 1 & 0 & 8 & 3 \\ 6 & 6 & 0 & 5 \end{bmatrix}$$
 and let $\vec{b} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Use Cramer's Rule
and the fact that $detA = 40$ to find x_1 where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a solution

of the system of equations $A\vec{x} = \vec{b}$.

$$\text{By Cramer's Rule, } x_1 = \frac{\det \begin{bmatrix} 5 & 8 & -2 & 12 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 8 & 3 \\ 0 & 6 & 0 & 5 \end{bmatrix}}{\det A} = \frac{\det \begin{bmatrix} 5 & 8 & -2 & 12 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 8 & 3 \\ 0 & 6 & 0 & 5 \end{bmatrix}}{40}.$$

$$det \begin{bmatrix} 5 & 8 & -2 & 12 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 8 & 3 \\ 0 & 6 & 0 & 5 \end{bmatrix} = det \begin{bmatrix} 5 & 8 & -2 & 12 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 8 & 3 \\ 0 & 0 & 0 & 11 \end{bmatrix} = 880.$$
 Therefore, $x_1 = \frac{880}{40} = 22.$

 $x_1 = 22$

9. For each of the following subsets of \mathbb{R}^3 , determine if the subset is a subspace of \mathbb{R}^3 .

(a)
$$W = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

Since for each vector in W, the first and third component are different, the zero vector is not in W so W is not a subspace of \mathbb{R}^3 .

 $\Box W$ is a subspace of \mathbb{R}^3 . $\Box W$ is not a subspace of \mathbb{R}^3 .

(b)
$$U = \left\{ \begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}.$$

Since $\vec{0} = \begin{bmatrix} 0 \\ 2 \cdot 0 \\ 0 \end{bmatrix}$, $\vec{0} \in U$. If $\vec{x} = \begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y \\ 2y \\ 0 \end{bmatrix}$
are elements of U , then $\vec{x} + \vec{y} = \begin{bmatrix} x + y \\ 2(x+y) \\ 0 \end{bmatrix} \in U.$ If
 $\vec{x} = \begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix} \in U$ and c is a scalar, then $c\vec{x} = \begin{bmatrix} cx \\ 2(cx) \\ 0 \end{bmatrix} \in U.$ Therefore, U is a subspace of \mathbb{R}^3 .

 $\boxtimes U$ is a subspace of \mathbb{R}^3 . $\Box U$ is not a subspace of \mathbb{R}^3 .

10. Let $M_{2\times3}$ denote the vector space of 2×3 matrices, and let H the subset of $M_{2\times3}$ consisting of the 2×3 matrices having at least one entry which is 0. Determine whether or not H is a subspace of $M_{2\times3}$ and give a reason for your answer.

It is not the case that the sum of elements of H is always an element of H. For example, if $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, then A and B are both elements of H but A+Bis not an element of H. Therefore, H is not a subspace of $M_{2\times 3}$.

 \Box H is a subspace of \mathbb{R}^3 .

 \boxtimes H is not a subspace of \mathbb{R}^3 .

Reason:

The sum of elements in H is not necessarily in H.