

The Cohomology Ring of Weight Varieties and Polygon Spaces

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We use a theorem of S. Tolman and J. Weitsman (The cohomology rings of Abelian symplectic quotients, math. DG/9807173) to find explicit formulæ for the rational cohomology rings of the symplectic reduction of flag varieties in \mathbb{C}^n , or generic coadjoint orbits of $SU(n)$, by (maximal) torus actions. We also calculate the cohomology ring of the moduli space of n points in $\mathbb{C}P^k$, which is isomorphic to the Grassmannian of k planes in \mathbb{C}^n , by realizing it as a degenerate coadjoint orbit. © 2001 Academic Press

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1. INTRODUCTION

For M a manifold with a Hamiltonian T action and moment map $\phi: M \rightarrow \mathfrak{t}^*$, the symplectic reduction is defined as

$$M//T(\mu) := \phi^{-1}(\mu)/T$$

for any regular value μ of ϕ . *Weight varieties* are a special case of symplectic reduction. Let $M = \mathcal{O}_\lambda$, a coadjoint orbit of a compact, semi-simple Lie group G through the point $\lambda \in \mathfrak{t}^*$, and consider the action of the maximal torus $T \subset G$ on \mathcal{O}_λ . If $G = SU(n)$, we identify the set of Hermitian matrices \mathcal{H} with \mathfrak{g}^* by $tr: A \rightarrow i \cdot \text{Trace}(A \cdot \cdot)$ for all $A \in \mathcal{H}$. Under this identification, we can think of λ as a matrix with real diagonal entries $(\lambda_1, \dots, \lambda_n)$, and \mathcal{O}_λ as an adjoint orbit of G through λ . The moment map for the T action on \mathcal{O}_λ takes a matrix to its diagonal entries. Thus $\mathcal{O}_\lambda//T(\mu)$ consists of Hermitian matrices with spectrum λ and diagonal entries μ , quotiented out

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by the action of diagonal matrices. The symplectic reduction $\mathcal{O}_\lambda//T(\mu)$ is a weight variety.

The generic coadjoint orbit of $SU(n)$ is symplectomorphic to the complete flag variety in \mathbb{C}^n with a symplectic structure given by the spectrum of the orbit. Degenerate coadjoint orbits are homeomorphic to flag varieties with various dimensions missing: for example if the spectrum consists of only two values, then the coadjoint orbit is homeomorphic to $Gr(k, n)$, the Grassmannian of k planes in \mathbb{C}^n . Then $\mathcal{O}_\lambda//T(\mu)$ is the symplectic reduction of a flag variety by a torus; however, the reduction depends on the symplectic structure on the flag variety.

Weight varieties have appeared in several different contexts. They were first termed as such by Knutson in [17] because of their relationship to the weight spaces of representations. Irreducible representations V of complex G are realized as the holomorphic sections of line bundles over the flag varieties. The dimension of the *weight spaces*, or the irreducible representations of T in V (specified by the weight of the T action on each isotypic component of V) is the *quantization* of the symplectic reduction of flag varieties [12]; hence these reductions were named weight varieties. Techniques to compute their Betti numbers have been developed by Kirwan [15] and Klyachko [16]. The advantage of the results in this article is that one obtains the cohomology ring structure. For Betti numbers, one may find that the calculations are straightforward (and hence programmable) using Theorems 1.1 and 1.2, but other methods are likely more efficient. In a small number of cases, the spaces have been found explicitly [13].

The symplectic reduction of the Grassmannian of k -planes in \mathbb{C}^n also arises as the moduli space of n points on $\mathbb{C}P^k$, as we see below; more details for the case when $k = 1$ can be found in the work of Klyachko [16], Hausmann and Knutson [18] and Kapovich and Millson [14]. The integer cohomology ring of the $k = 1$ polygon spaces was computed in [19].

The generic case we consider is the coadjoint orbit of $SU(n)$ of matrices with a specified set of distinct eigenvalues. Equivalently, it is the orbit through $\lambda \in \mathfrak{t}^* := Lie(T)^*$, where λ consists of n distinct eigenvalues $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\sum_1^n \lambda_i = 0$, and T is the $(n-1)$ -dimensional maximal torus of $SU(n)$. We order them such that $\lambda_1 > \dots > \lambda_n$. These coadjoint orbits are diffeomorphic to the complete flag manifold $Fl(n)$ in \mathbb{C}^n . The Grassmannian of k -planes in \mathbb{C}^n is diffeomorphic to the $SU(n)$ coadjoint orbit through $\nu \in \mathfrak{t}^*$ where $\nu = (\nu_1, \dots, \nu_1, \nu_2, \dots, \nu_2)$ consists of two distinct eigenvalues ν_1 and ν_2 . The dimension of the ν_1 eigenspace is k and that of the ν_2 eigenspace is $n-k$, so we have $k\nu_1 + (n-k)\nu_2 = 0$. We will write $Gr(k, n)_\nu$ to indicate these degenerate coadjoint orbits.

We use the notation λ_w to indicate the point $w\lambda w^{-1} \in \mathfrak{t}^*$ for any permutation w on n letters. This has the unfortunate consequence that $\lambda_w = (\lambda_{w^{-1}(1)}, \dots, \lambda_{w^{-1}(n)})$ but allows us to use left actions consistently

throughout the paper. Furthermore, let $\Delta(x, u) \in \mathbb{C}[x_1, \dots, x_n, u_1, \dots, u_n]$ be the polynomial

$$\Delta(x, u) = \prod_{i < j} (x_i - u_j).$$

$\Delta(x, u)$ is sometimes called the *determinant polynomial*.

Before we state the main theorems, we briefly introduce *divided difference operators*, which are explained in detail in Section 2.3.

DEFINITION 1.1. Let $f(x_i, x_{i+1})$ be a polynomial of variables x_i and x_{i+1} and possibly other variables. For each $1 \leq i \leq n$ the *divided difference operator* ∂_i associated to i acts on f as follows:

$$\partial_i f(x_i, x_{i+1}) = \frac{f(x_i, x_{i+1}) - f(x_{i+1}, x_i)}{x_i - x_{i+1}}.$$

These simple divided difference operators take polynomials of degree k to polynomials of degree $k - 1$. For any element $w \in S_n$, the permutation group on n letters, one can associate a divided difference operator ∂_w as follows. Write w as a product of simple transpositions $s_{i_1} \cdots s_{i_l}$ where s_{i_j} is the simple transposition that switches i_j and $i_j + 1$. For any such product where l is minimal, the composition

$$\partial_w := \partial_{i_1} \cdots \partial_{i_l}$$

is well defined (see [22]). We are now ready to state the main theorems of this article.

THEOREM 1.1. *Let \mathcal{O}_λ be a generic coadjoint orbit of $SU(n)$. The rational cohomology of $\mathcal{O}_\lambda // T(\mu)$ is isomorphic to the ring*

$$\frac{\mathbb{C}[x_1, \dots, x_n, u_1, \dots, u_n]}{(\prod_{i=1}^n (1 + u_i) - \prod_{i=1}^n (1 + x_i), \sum_{i=1}^n u_i, \partial_v \Delta(x, u_\tau))}$$

for all $v, \tau \in S_n$ such that $\sum_{i=k+1}^n \lambda_{v(i)} < \sum_{i=k+1}^n \mu_{\tau(i)}$ for some $k = 1, \dots, n - 1$. Here $\deg x_i = \deg u_i = 2$, and $\prod (1 + u_i) - \prod (1 + x_i)$ is the graded difference of symmetric functions in the x_i s and u_i s.

There is a similar statement for the degenerate case of the Grassmannian, in which there are only two distinct eigenvalues.

THEOREM 1.3. *The rational cohomology of $Gr(k, n)_v // T(\mu)$ is isomorphic to the ring*

$$\frac{\mathbb{C}[\sigma_i(x_1, \dots, x_k), \sigma_i(x_{k+1}, \dots, x_n), u_1, \dots, u_n]}{(\sigma_i(x_1, \dots, x_n) - \sigma_i(u_1, \dots, u_n), \sum_{i=1}^n u_i, \partial_v \Delta(x, u_\tau))}$$

for all v, τ such that $\sum_{i=k+1}^n \lambda_{v(i)} < \sum_{i=k+1}^n \mu_{\tau(i)}$ for some k , and $\partial_v \Delta(x, u_\tau)$ is symmetric in x_1, \dots, x_k and in x_{k+1}, \dots, x_n . Here σ_i are the symmetric polynomials in the indicated variables and $\deg x_i = \deg u_i = 2$. Note that $\sigma_i(x_1, \dots, x_n) - \sigma_i(u_1, \dots, u_n)$ for all i is equivalent to $\prod_{i=1}^n (1 + u_i) - \prod_{i=1}^n (1 + x_i)$.

The forgetful map $Fl(n) \rightarrow Gr(k, n)$ which “remembers” only the k -planes of each flag is a T -equivariant map which induces an injection on equivariant cohomology:

$$H_T^*(Gr(k, n)) \hookrightarrow H_T^*(Fl(n)). \tag{1}$$

The classes in $H_T^*(Fl(n))$ which are symmetric in x_1, \dots, x_k and in x_{k+1}, \dots, x_n are precisely those in the image of the map (1).

Theorem 1.2 allows one to compute the cohomology of the moduli space of n points in $\mathbb{C}P^{k-1}$, as the reduction of the Grassmannian is isomorphic to this moduli space [10], [9]. In [18], Knutson and Hausmann observe that this Gelfand–MacPherson correspondence is just a dual pair symplectic reduction. The Grassmannian $Gr(k, n)$ is realized as a symplectic reduction of \mathbb{C}^{nk} by a $U(k)$ action on the right. Then by reducing in stages we have

$$\begin{aligned} Gr(k, n) // T^n &= (\mathbb{C}^{nk} // U(k)) // T^n = \mathbb{C}^{nk} // (T^n \times U(k)) \\ &= (\mathbb{C}^{nk} // T^n) // U(k) = \prod_{i=1}^n \mathbb{C}P^{k-1} // U(k). \end{aligned}$$

This last space is exactly the moduli space of n points in $\mathbb{C}P^k$. The group $U(k)$ does not act effectively on $\prod_{i=1}^n \mathbb{C}P^{k-1}$. The center, consisting of scalar matrices, acts trivially. We quotient this S^1 out and find

$$Gr(k, n) // T = \prod_{i=1}^n \mathbb{C}P^{k-1} // PU(k),$$

where T is the $(n - 1)$ -dimensional torus used in Theorem 1.3. Putting in the symplectic structure, we make the following statement.

COROLLARY 1.4. *Let \mathcal{M} be the moduli space of n points in $\mathbb{C}P^{k-1}$, realized as above by a symplectic reduction as follows:*

$$\begin{aligned} \mathcal{M} &= (\mathbb{C}^{nk} // U(k))(aI) // T^n(\mu_1 + a, \mu_2 + a, \dots, \mu_n + a) \\ &= Gr(k, n)_{(v_1, \dots, v_1, v_2, \dots, v_2)} // T(\mu_1, \dots, \mu_n). \end{aligned}$$

where $a = (v_1 - v_2)$ and I is the identity matrix in $\mathfrak{u}^*(n)$. Then $H^*(\mathcal{M})$ is isomorphic to the ring

$$\frac{\mathbb{C}[\sigma_i(x_1, \dots, x_k), \sigma_i(x_{k+1}, \dots, x_n), u_1, \dots, u_n]}{(\sigma_i(x_1, \dots, x_n) - \sigma_i(u_1, \dots, u_n), \sum_{i=1}^n u_i, \partial_v A(x, u_\tau))}$$

for all v, τ such that $\sum_{i=k+1}^n \lambda_{v(i)} < \sum_{i=k+1}^n \mu_{\tau(i)}$ for some k , and $\partial_v A(x, u_\tau)$ is symmetric in x_1, \dots, x_k and in x_{k+1}, \dots, x_n .

Remark 1.1. The equivariant Chern class of the tangent bundle $TM \rightarrow M$ descends to the ordinary (total) Chern class of $M//T$ under symplectic reduction by a torus. We find the Chern class of the tangent bundle to $Fl(n) \cong \mathcal{O}_\lambda$. We note that $c(\mathbb{C}P^n) = (1-x)^{n+1}$, where x is the first Chern class of the tautological line bundle S over CP^n (see [6]). Using an inductive argument on the fibration $Fl(n-1) \hookrightarrow Fl(n) \rightarrow \mathbb{C}P^n$, one can show that in the basis used above, the Chern class of \mathcal{O}_λ is $(1-x_1)^n (1-x_2)^{n-1} \cdots (1-x_{n-1})^2$. Then this is also the total Chern class of the weight varieties $\mathcal{O}_\lambda // T(\mu)$.

There are two essential facts that come into play in the results presented here. For M a symplectic manifold with a Hamiltonian torus action, there is a restriction map in equivariant cohomology from M to the μ -level set $\phi^{-1}(\mu)$ of the moment map ϕ . The (rational) equivariant cohomology of the level set is equal to the regular cohomology of the reduced space $M//T(\mu) := \phi^{-1}(\mu)/T$. The first theorem is that the resulting map is a surjection.

THEOREM 1.3 (Kirwan). *Let M be a Hamiltonian T space with moment map ϕ and $\mu \in \mathfrak{t}^*$ a regular value of ϕ . Then the map induced by restriction to the level set $\phi^{-1}(\mu)$*

$$\kappa_\mu: H_T^*(M) \rightarrow H^*(M//T(\mu))$$

is a surjection.

Second, the restriction map in equivariant cohomology induced by the inclusion of the fixed point set M^T into M is an injection.

THEOREM 1.4 (Chang-Skjelbred, Kirwan). *Let M be a Hamiltonian T space with fixed point set M^T . The natural map*

$$r^*: H_T^*(M) \hookrightarrow H_T^*(M^T)$$

is an inclusion.

Note how different this is from ordinary cohomology. If M has isolated fixed points, for example, $H^*(M^T)$ is zero except in degree 0, yet $H^*(M)$ may have cohomology in higher degree.

Theorem 1.4 suggests the following definition.

DEFINITION 1.2. Let $\alpha \in H_T^*(M)$ be an equivariant cohomology class on a compact Hamiltonian T space with fixed point set M^T . Define the *support* of α to be

$$\text{supp } \alpha = \{C \text{ connected component of } M^T \mid r_C^*(\alpha) \neq 0\}$$

where $r_C^*: H_T^*(M) \rightarrow H_T^*(C)$ is the restriction to the equivariant cohomology of the fixed component C .

By Theorem 1.3, the cohomology of the symplectic reduction can be computed as the quotient of the equivariant cohomology $H_T^*(M)$ by the kernel of the Kirwan map κ . Theorem 1.4 indicates that the kernel may be generated by cohomology classes which have certain properties restricted to the fixed point set. This line of reasoning was exploited by Tolman and Weitsman who described the kernel κ_μ [23].

THEOREM 1.5 (Tolman–Weitsman). *Let M be a compact symplectic manifold with a Hamiltonian T action. Let $\phi: M \rightarrow \mathfrak{t}^*$ be a moment map such that μ is a regular value. For any $\xi \in \mathfrak{t}$, define*

$$M_\xi^\mu := \{m \in M \mid \langle \phi(m), \xi \rangle \leq \langle \mu, \xi \rangle\}$$

and

$$K_\xi := \{\alpha \in H^*(M) \mid \text{supp } \alpha \subset M_\xi^\mu\}.$$

Then the kernel of the natural map $\kappa_\mu: H_T^(M) \rightarrow H^*(M//T(\mu))$ is the ideal $\langle K \rangle$ generated by*

$$K := \bigcup_{\xi \in \mathfrak{t}} K_\xi.$$

The classes in the kernel are generated by classes which have non-zero restriction to fixed points which (under the moment map) lie entirely to

one side of a hyperplane ξ_μ^\perp through μ in \mathfrak{t}^* . Because M is compact, only a finite number of hyperplanes will be necessary.

The contribution of this article is the application of the work of Tolman and Weitsman to the case where M is a coadjoint orbit of $SU(n)$. Here there is an explicit description of a generating set of classes for the ideal $\langle K \rangle$, which allows one to actually compute the cohomology ring of weight varieties. These classes are represented by double Schubert polynomials, permuted by the Weyl group and satisfying certain properties. Double Schubert polynomials were first introduced by Lascoux and Schützenberger [21], [22]. As a corollary, for $SU(n)$ coadjoint orbits, the only vectors $\xi \in \mathfrak{t}$ needed for Theorem 1.5 are fundamental weights and their permutations (Corollary 4.1). Equivalently, the kernel of the Kirwan map κ_μ is generated by classes which restrict to zero on one side of hyperplanes $\xi^\perp := \text{ann}(\xi)$, translated to contain μ , parallel to codimension-one walls of the moment polytope.

2. THE T -EQUIVARIANT COHOMOLOGY OF FLAG MANIFOLDS AND GRASSMANNIANS

There are two descriptions of the equivariant cohomology of the flag manifold that we will use here. First is a presentation of the ring as a quotient of a polynomial ring by relations. This is due in the non-equivariant setting to Borel [4] and can be found in [6]. The equivariant version can be obtained from the standard computation by using the Borel construction: By definition, $H_T^*(Fl(\mathbb{C}^n)) := H^*(Fl(\mathbb{C}^n) \times_T ET)$ where ET is a universal T bundle, and note that $Fl(\mathbb{C}^n) \times_T ET$ is a flag bundle over the classifying space $BT := ET/T$. Using standard methods, one computes the cohomology of this bundle. Details can be found in [11], [7].

The second description is due to Arabia [1], and says that there is a basis for $H_T^*(G/T)$ as a module over $H_T^* := H_T^*(pt)$ which has certain properties that we expand upon below. The connection between these two pictures in the case that $G = SU(n)$ is provided by double Schubert polynomials. We show by Theorem 2.3 and Theorem 2.5 that double Schubert polynomials, and their corresponding classes under a quotient map, have the properties specified by Arabia. Hence the linear basis can be expressed in the presentation due to Bernstein, Gelfand, and Gelfand.

2.1. Presentations of $H_T^*(Fl(\mathbb{C}^n))$ and $H_T^*(Gr(k, n))$

The T^n action on $Fl(\mathbb{C}^n)$ which is induced by the action on \mathbb{C}^n of weight 1 on each of n copies of \mathbb{C} is not effective: a diagonal $S^1 \subset T^n$ fixes $Fl(\mathbb{C}^n)$. We quotient by this circle and let T be the $(n-1)$ -torus which acts effectively.

THEOREM 2.1. *Let T be the maximal $(n - 1)$ -dimensional torus in $SU(n)$. The T -equivariant cohomology of the generic $SU(n)$ -coadjoint orbit is*

$$H_T^*(Fl(\mathbb{C}^n)) = \frac{\mathbb{C}[x_1, \dots, x_n, u_1, \dots, u_n]}{(\prod_{i=1}^n (1 + u_i) - \prod_{i=1}^n (1 + x_i), \sum_{i=1}^n u_i)}, \quad i = 1, \dots, n$$

where $\deg u_i = \deg x_i = 2$, and the u_i are the image of the classes from the module structure $H_T^* \rightarrow H_T^*(Fl(\mathbb{C}^n))$.

It is straightforward to see that $H_T^*(Fl(\mathbb{C}^n)) = H^*(Fl(E))$, the flags of the bundle $E := \mathbb{C}^n \times_T ET \rightarrow BT$ for a certain action of T on \mathbb{C}^n . $Fl(E)$ can be canonically realized as a tower of projective bundles over BT

$$Fl(E) = \mathbb{P}(Q_{n-1}) \xrightarrow{\pi_n} \dots \xrightarrow{\pi_3} \mathbb{P}(Q_1) \xrightarrow{\pi_2} \mathbb{P}(E) \xrightarrow{\pi_1} BT$$

where $Q_1 := \pi_1^* E/S_1$ for S_1 the tautological line bundle over $\mathbb{P}(E)$. Thus $\pi_1^* E = Q_1 \oplus S_1$ and the bundle $Q_1 \rightarrow \mathbb{P}(E)$ can be projectivized to form $\pi_2: \mathbb{P}(Q_1) \rightarrow \mathbb{P}(E)$. We repeat this process to obtain $Q_k := \pi_k^* \dots \pi_1^* E/S_1 \oplus \dots \oplus S_k$ where S_k is the tautological line bundle over $\mathbb{P}(Q_{k-1})$. The pullback of E to $Fl(E)$ canonically splits into a sum of line bundles

$$\begin{array}{ccc} \pi^* E = S_1 \oplus \dots \oplus S_n & \longrightarrow & E \\ \downarrow & & \downarrow \\ Fl(E) & \xrightarrow{\pi} & BT. \end{array}$$

By definition, $x_i = c_1(S_i)$. It follows that

$$H_T^*(Fl(E)) = H^*(BT)[x_1, \dots, x_n] \Big/ \prod_{i=1}^n (1 + x_i) = c(E).$$

Using the splitting principle one obtains that the total Chern class of E is $\prod_{i=1}^n (1 + u_i)$, where u_i is the first Chern class of the tautological line bundle over the i th copy of S^1 in T under a choice of decomposition $T = S^1 \times \dots \times S^1$. Then restricting to the $n - 1$ dimensional torus that acts effectively on $Fl(E)$, we get the desired result.

THEOREM 2.2. *Let T be the maximal $(n - 1)$ -dimensional torus in $SU(n)$. The T -equivariant cohomology of the Grassmannian of complex k -planes in \mathbb{C}^n is a subring of $H_T^*(Fl(\mathbb{C}^n))$ and can be written*

$$H_T^*(Gr(k, n)) = \frac{\mathbb{C}[\sigma_i(x_1, \dots, x_k), \sigma_i(x_{k+1}, \dots, x_n), u_1, \dots, u_n]}{(\sigma_i(x_1, \dots, x_n) - \sigma_i(u_1, \dots, u_n), \sum_{i=1}^n u_i)},$$

where σ_i is the i th symmetric function in the indicated variables, $i = 1, \dots, n$. The Chern classes x_i and u_i are the same as those in Theorem 2.1.

We now explore the relationship between this quotient description of the equivariant cohomology, and the description as a subring of the equivariant cohomology of the fixed point set.

2.2. *The Restriction of Equivariant Cohomology to Fixed Points*

By Theorem 1.4, the T -equivariant cohomology of $Fl(\mathbb{C}^n)$ can be embedded in a direct sum of polynomial rings. We calculate this restriction explicitly using geometric means. A purely algebraic proof can be found in [7]. In [3] Billey finds a simple formula for the restriction of Kostant polynomials (generalized double Schubert polynomials) to the fixed point set.

The fixed point set $Fl(\mathbb{C}^n)^T$ is indexed by the Weyl group $W = N(T)/T$. The T action on $V = \mathbb{C}^n$ splits V into a sum of 1-dimensional vector spaces, or lines which we order and call l_1, \dots, l_n . The fixed points of T on $Fl(V)$ are the flags which can be written $\langle l_{i_1} \rangle \subset \langle l_{i_1}, l_{i_2} \rangle \subset \dots \subset \langle l_{i_1}, \dots, l_{i_n} \rangle = V$. We call $\langle l_1 \rangle \subset \langle l_1, l_2 \rangle \subset \dots \subset \langle l_1, \dots, l_n \rangle = V$ the base flag and label the fixed points by the corresponding permutation in W :

$$p_w := \langle l_{w(1)} \rangle \subset \langle l_{w(1)}, l_{w(2)} \rangle \subset \dots \subset \langle l_{w(1)}, \dots, l_{w(n)} \rangle.$$

The restriction map from the description in Theorem 2.1 to the fixed point set is as follows:

THEOREM 2.3. *Let $p_w \in Fl(V)$ be in the fixed point set $Fl(V)^T$ as above. The inclusion $r_w: p_w \rightarrow Fl(V)$ induces a restriction*

$$r_w^*: H_T^*(Fl(V)) \rightarrow H_T^*(p_w) = S(\mathfrak{t}^*) = \mathbb{C}[u_1, \dots, u_n] \tag{2}$$

such that $r_w^*: x_i \mapsto u_{w(i)}$ and $r_w^*: u_i \mapsto u_i$, where x_i and u_i , $i = 1, \dots, n$ are the generators of the equivariant cohomology in Theorem (2.1). In particular, $r: Fl(V)^T \hookrightarrow Fl(V)$ induces a map

$$r^*: H_T^*(Fl(V)) \rightarrow H_T^*(Fl(V)^T) = \bigoplus_{p \in W} \mathbb{C}[u_1, \dots, u_n]$$

whose further restriction to each component in the direct sum is r_w^* .

Proof. The classes u_i come from the module structure of $H_T^*(M)$ over H_T^* . For any $p \hookrightarrow M$, the induced map $H_T^*(M) \rightarrow H_T^*(p)$ is a module map, and hence $r_w^* u_i = u_i$ for all w, i . A fixed point $p \in M$ in the Borel construction corresponds to a fixed copy of $BT \cong p \times_T ET$ in $M \times_T ET$. We can restrict the x_i to the fixed points of $Fl(V)$, and then use the Borel construction to make the restriction equivariant.

We first compute $r_w^*(x_1)$ for any $w \in W$, and proceed inductively to obtain $r_w^*(x_i)$ for all i . Let $E := V \times_T \mathbb{C}^n$ and note that $Fl(E) = Fl(V) \times_T ET$. The class $x_i = c_1(S_i) \in H^*(Fl(E))$ by definition. The equivariant restriction to p_w is the usual restriction to $p_w \times_T ET$ in $Fl(E)$. S_1 is the (pullback of) the tautological line bundle over $\mathbb{P}(E)$. Under the projection to $\mathbb{P}(E)$, $p_w \times_T ET$ projects to $l_{w(1)} \times_T ET$, or a section s_w of $\mathbb{P}(E) \rightarrow BT$. Then

$$r_w^*(x_1) = c_1(S_1)|_{l_{w(1)} \times_T ET}.$$

As a line bundle, $l_{w(1)} \times_T ET = s_w^* S_1$ from the commutative diagram

$$\begin{array}{ccc} s_w^* S_1 & \longrightarrow & S_1 \\ \downarrow & & \downarrow \\ BT & \xrightarrow{s_w} & \mathbb{P}(E). \end{array}$$

As E splits into

$$E = \bigoplus_{i=1}^n L_i,$$

with $L_i = l_i \times_{T_i} ET_i$ and $T_i \cong S^1$ acts on l_i with weight one and on l_j trivially for $j \neq i$, we have

$$s_w^* S_1 = l_{w(1)} \times_T ET = L_{w(1)}$$

so that

$$r_w^*(x_1) = c_1(L_{w(1)})$$

by naturality of the pullback map. We note that by definition, $u_i = c_1(L_i)$ and hence

$$r_w^*(x_1) = c_1(S_1)|_{l_{w(1)} \times_T ET} = c_1(L_{w(1)}) = u_{w(1)}.$$

We continue inductively: the point p_w also specifies a two-dimensional fixed subspace $\langle l_{w(1)}, l_{w(2)} \rangle$ of V containing $l_{w(1)}$. Under the Borel construction, this two-space is a section in the projectivization $\mathbb{P}(Q_1)$ of the quotient $Q_1 = \pi_1^* E/S_1$, or the restriction of its tautological line bundle S_2 to the copy of BT that is the image of this section. Then

$$r_w^*(x_2) = c_1(S_2)|_{l_{w(2)} \times_T ET} = u_{w(2)}.$$

It is clear how to proceed from here. For each x_i , there is a projection from p_w to the corresponding i -space, which can (after crossing with ET and modding out by T) be realized as a line in S_i . We obtain

$$r_w^*(x_i) = c_1(S_i)|_{L_{w(i)} \times_T ET} = u_{w(i)}$$

which concludes the proof. ■

An example of carrying out such a restriction may be useful to the reader.

EXAMPLE 2.1. Let $n=3$, and $\alpha = (x_1 - u_1)(x_1 - u_3)$ be a T -equivariant cohomology class on \mathcal{O}_λ , a generic coadjoint orbit of $SU(3)$. There are six fixed points of T acting on \mathcal{O}_λ , labeled by permutations in S_3 . Let $\alpha|_w$ indicate the restriction of α to λ_w . In one-line notation, the restrictions are:

$$\begin{aligned} \alpha|_{[123]} &= 0 & \alpha|_{[213]} &= (u_2 - u_1)(u_2 - u_3) \\ \alpha|_{[132]} &= 0 & \alpha|_{[231]} &= (u_2 - u_1)(u_2 - u_3) \\ \alpha|_{[312]} &= 0 & \alpha|_{[321]} &= 0. \end{aligned}$$

2.3. Divided Difference Operators and Double Schubert Polynomials

The divided difference operators mentioned in Section 1 were introduced by Bernstein, Gelfand and Gelfand [2] and independently by Demazure [8]. They take polynomials to polynomials, and send to 0 anything symmetric in the variables of the operator. For this reason, they act on the (ordinary) cohomology of G/T , with T the maximal torus of a compact Lie group G . These operators act on the T -equivariant cohomology of G/T as well; by applying divided difference operators to a certain equivariant cohomology class of degree equal to the dimension of G/T , one can generate a linear basis for $H_T^*(G/T)$. From a combinatorial perspective, this is equivalent in the $G = SU(n)$ case to generating double Schubert polynomials by the divided difference operators applied to a “determinant polynomial”. This was first carried out by Lascoux and Schützenberger [20], see also [22]. In this section we make explicit the connection between the combinatorics and the equivariant cohomology. By allowing the Weyl group to act on the cohomology ring, we obtain *permuted* double Schubert polynomials, which provide Weyl group many linear bases for $H_T^*(G/T)$ as a module over H_T^* , as we will describe in Section 2.4.

DEFINITION 2.1. Let $f(x_i, x_{i+1})$ be a polynomial of variables x_i and x_{i+1} and possibly other variables. For each $1 \leq i \leq n$ we define the *divided difference operator* ∂_i as follows:

$$\partial_i f(x_i, x_{i+1}) = \frac{f(x_i, x_{i+1}) - f(x_{i+1}, x_i)}{x_i - x_{i+1}}.$$

The resulting function $\partial_i f$ is also a polynomial.

Let W be the Weyl group for $SU(n)$, i.e. $W = N(T)/T$. Then W is isomorphic to S_n , the permutation group on n letters. Furthermore, W is generated by *simple transpositions* s_i which interchange i and $i+1$. Any element $w \in W$ can thus be written as a product $w = s_{i_1} s_{i_2} \cdots s_{i_l}$. Whenever w is written with l minimum, we call the expression a *reduced word* for w . For any such reduced expression, we can define the operator

$$\partial_{s_{i_1} s_{i_2} \cdots s_{i_l}} = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_l}.$$

It turns out that the resulting operator is independent of the choice of reduced word for w (see [22]). We can thus define the divided difference operator associated to the element $w \in W$ as

$$\partial_w = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_l}$$

for any reduced word expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$.

We now consider the following set of polynomials generated from one polynomial by the divided difference operators ∂_w and permuted by the Weyl group.

DEFINITION 2.2. The *determinant polynomial* $\Delta \in \mathbb{C}[x_1, \dots, x_n, u_1, \dots, u_n]$ is defined to be

$$\Delta(x, u) = \prod_{i < j} (x_i - u_j).$$

The *permuted double Schubert polynomials* \mathfrak{S}_w^τ are defined by successive application of divided difference operators to Δ , as follows. The identity double Schubert polynomials are

$$\mathfrak{S}_w^{id} := \partial_{w^{-1}} \Delta.$$

The permuted double Schubert polynomials are defined from the identity ones,

$$\mathfrak{I}_w^\tau(x, u) := \mathfrak{I}_{\tau^{-1}w}^{id}(x, u_\tau),$$

where u_τ indicates the permutation of the u variables by τ .

EXAMPLE 2.2. We compute $\mathfrak{I}_w^\tau(x, u)$ for $n = 3$, with $\tau = [213]$ and $w = [231] = s_1 s_2$ in one-line notation. We have $\Delta(x, u) = (x_1 - u_2)(x_1 - u_3)(x_2 - u_3)$, and $\Delta(x, u_\tau) = (x_1 - u_1)(x_1 - u_3)(x_2 - u_3)$. Then

$$\begin{aligned} \mathfrak{I}_w^\tau(x, u) &= \mathfrak{I}_{[132]}^{[123]}(x, u_{[213]}) \\ &= \partial_2 \Delta(x, u_{[213]}) \\ &= \partial_2 (x_1 - u_1)(x_1 - u_3)(x_2 - u_3) \\ &= (x_1 - u_1)(x_1 - u_3). \end{aligned}$$

A couple comments are in order here.

1. For those familiar with double Schubert polynomials and determinant polynomials, we note that in [22] the determinant polynomial is what we have called $\Delta(x_{w_0}, u)$ for w_0 the long word in W . Double Schubert polynomials $\mathfrak{S}_w(x, u)$ are equivalent to $\mathfrak{I}_w^{w_0}$.

2. In this notation, the polynomials $\partial_w(x, u_\tau) = \mathfrak{I}_{\tau w^{-1}}^\tau(x, u)$; equivalently, $\mathfrak{I}_w^\tau = \partial_{w^{-1}\tau}(x, u_\tau)$.

3. While these definitions may seem to have an ungainly number of inverses, the resulting geometric interpretation has a simple statement; see Theorem 2.4.

Under the quotient map

$$\mathbb{C}[x_1, \dots, x_n, u_1, \dots, u_n] \rightarrow H_T^*(G/T)$$

the polynomials \mathfrak{I}_v^τ descend to cohomology classes; we sloppily refer the classes themselves as \mathfrak{I}_v^τ .

2.3.1. *The Bruhat order and permuted schubert varieties.* Let $G^\mathbb{C}$ be the complexification of a compact Lie group G , and let $B \subset G^\mathbb{C}$ be a Borel subgroup. For example, for $G = SU(n)$, $G^\mathbb{C} = Sl(n, \mathbb{C})$ and one choice of B is upper-triangular matrices. Then $G^\mathbb{C}/B \cong G/T$.

The space $G^\mathbb{C}/B$ is composed of even-real-dimensional Schubert cells indexed by elements in the Weyl group $W = N(T^\mathbb{C})/T^\mathbb{C}$

$$C_w := BwB/B, \quad w \in W.$$

Technically, one needs to choose a lift of w in $N(T^C)$ but the cell is independent of this choice, and so it is standard to consider $w \in W$. The closures of these cells are *Schubert varieties*

$$X_w := \overline{BwB/B}$$

and were shown to generate the homology of G^C/B [2]. We define the *permuted Schubert cells* as

$$C_w^\tau := \tau B \tau^{-1} w B / B$$

and the *permuted Schubert varieties*

$$X_w^\tau := \overline{\tau B \tau^{-1} w B / B}.$$

Note that these are $\tau B \tau^{-1}$ -invariant varieties which also generate the homology of G^C/B .

EXAMPLE 2.3. The Schubert variety X_{w_0} consists of all of G^C/B , whereas the variety X_{id} consists of one point (the identity coset). Similarly, $X_{\tau w_0}^\tau$ is all of G^C/B and X_τ^τ consists of just a point: the permutation τ .

The definition of these varieties suggest a partial ordering on them, and hence on the elements of W that index them. The Bruhat order is defined:

$$v \leq w \quad \text{if and only if} \quad C_v \subset X_w.$$

Similarly, we define a “permuted” ordering:

$$v \leq_\tau w \quad \text{if and only if} \quad C_v^\tau \subset X_w^\tau.$$

The simple relation between these two is that

$$v \leq_\tau w \quad \text{if and only if} \quad \tau^{-1}v \leq \tau^{-1}w.$$

Chevalley proved that the Bruhat ordering is equivalent to the following.

DEFINITION 2.3. We say that $v \leq w$ in the *Bruhat order* if and only if, for all reduced word expressions $w = t_1 \cdots t_l$, there is a subword $v = t_{i_1} \cdots t_{i_k}$ with $i_1 < \cdots < i_k$ which is a reduced word expression for v . We say that $v \leq_\tau w$ in the *permuted Bruhat order* if and only if $\tau^{-1}v \leq \tau^{-1}w$.

Lastly, we note that the T action on these varieties has fixed points:

$$X_w^T = \{v \in W : v \leq w\}$$

and more generally,

$$(X_w^\tau)^T = \{v \in W : v \leq_\tau w\}.$$

We now show a fundamental relation between the permuted double Schubert polynomials, viewed as cohomology classes, and the permuted Schubert varieties.

THEOREM 2.4. *The support of the equivariant cohomology class \mathfrak{F}_v^τ is the permuted Schubert variety X_v^τ .*

EXAMPLE 2.4. This theorem states that

$$\begin{aligned} \text{supp } \mathfrak{F}_{[231]}^{[213]} &= \{w \in W : w \leq_{[213]} [231]\} \\ &= \{w \in W : [213]^{-1} w \leq [213]^{-1} [231] = [132]\} \\ &= \{w \in W : [213]^{-1} w = [123] \text{ or } [132]\} \\ &= \{w = [213][123] = [213] \text{ or } w = [213][132] = [231]\}. \end{aligned}$$

In Examples 2.2 and 2.1, where we showed that $\mathfrak{F}_{[231]}^{[213]} = (x_1 - u_1)(x_1 - u_3)$ has support on $\{[213], [231]\}$, as expected.

Proof. We first prove that $\text{supp } \mathfrak{F}_v^{id} = (X_v^{id})^T$ using Theorem 2.3, and then the theorem will follow easily from the definition of the permuted double Schubert polynomials. Recall from Theorem 2.3 that the restrictions $u_i|_w = u_i$ and $x_i|_w = u_{w(i)}$ for all $w \in W$. We use induction on $l(w)$. For $w = id = [1 \ 2 \ \dots \ n]$, clearly the polynomial $\mathfrak{F}_{id}^{id} = \Delta$ restricts to zero at every point except λ_{id} . Suppose now that we have $\text{supp } \mathfrak{F}_w^{id} = (X_w^{id})^T$ for any w with $l(w) = l - 1$. Let $l(v) = l$, and a reduced word expression for v be $v = s_{i_1} s_{i_2} \dots s_{i_l}$. Let $w = v s_{i_l}$. Then restricted to λ_z we have

$$\begin{aligned} \mathfrak{F}_v^{id}|_{\lambda_z} &= \partial_{v^{-1}} \Delta|_{\lambda_z} = \partial_{i_l} \dots \partial_{i_1} \Delta|_{\lambda_z} = \partial_{i_l} \mathfrak{F}_w^{id}|_{\lambda_z} \\ &= \frac{\mathfrak{F}_w^{id}(x, u) - \mathfrak{F}_w^{id}(x s_{i_l}, u)}{x_{i_l} - x_{i_l+1}} \Big|_{\lambda_z} \\ &= \frac{\mathfrak{F}_w^{id}(u_z, u) - \mathfrak{F}_w^{id}(u_{z s_{i_l}}, u)}{u_{z(i_l)} - u_{z(i_l+1)}}. \end{aligned}$$

Suppose that $z \notin (X_v^{id})^T$. Then $z \notin (X_w^{id})^T$, and since $l(w) = l-1$, by our inductive hypothesis the restriction above is

$$\begin{aligned} \mathfrak{F}_v^{id}|_{\lambda_z} &= -\frac{\mathfrak{F}_w^{id}(u_{zs_{i_l}}, u)}{u_{z(i_l)} - u_{z(i_{l+1})}} \\ &= -\frac{r_{zs_{i_l}}^* \mathfrak{F}_w^{id}(x, u)}{u_{z(i_l)} - u_{z(i_{l+1})}} \end{aligned}$$

which is zero if $zs_{i_l} \notin (X_w^{id})^T$.

Suppose then that $zs_{i_l} \in (X_w^{id})^T$. Then if $z < zs_{i_l}$, we have $z \in (X_w^{id})^T$, a contradiction. If $z > zs_{i_l}$, then s_{i_l} increases the length of both zs_{i_l} and w . But then $zs_{i_l} \in (X_w^{id})^T$ implies $z \in (X_w^{id})^T$, again a contradiction.

We have shown that the restriction $\mathfrak{F}_v^{id}|_{\lambda_z}$ is zero unless $z \in (X_v^{id})^T$. It is left to show that $\mathfrak{F}_v^\tau|_{\lambda_z} = 0$ unless $z \in (X_v^\tau)^T$. By definition, $\mathfrak{F}_v^\tau(x, u) := \mathfrak{F}_{\tau^{-1}v}^{id}(x, u_\tau)$, and we showed that

$$\text{supp } \mathfrak{F}_{\tau^{-1}v}^{id}(x, u) = (X_{\tau^{-1}v}^{id})^T = \{w \in W : w \leq \tau^{-1}v\}.$$

Then, permuting the u 's by τ we obtain

$$\begin{aligned} \text{supp } \mathfrak{F}_v^\tau &= \text{supp } \mathfrak{F}_{\tau^{-1}v}^{id}(x, u_\tau) \\ &= \{\tau w \in W : w \leq \tau^{-1}v\} = \{w \in W : \tau^{-1}w \leq \tau^{-1}v\} \\ &= (X_v^\tau)^T. \quad \blacksquare \end{aligned}$$

2.4. A Linear Basis of $H_T^*(G/T)$

In [1] Arabia shows that there is a basis of $H_T^*(G/T)$ as a module over H_T^* with certain defining properties. We use Theorem 2.4 and Arabia's methods to show that \mathfrak{F}_w^τ satisfy these properties for $G = SU(n)$. It follows that, for any $\tau \in W$, $\alpha \in H_T^*(Fl(\mathbb{C}^n))$,

$$\alpha = \sum_{w \in W} a_w^\tau \mathfrak{F}_w^\tau$$

where $a_w^\tau \in H_T^*$.

THEOREM 2.5. *The classes \mathfrak{F}_v^τ have the (defining) properties that*

1. *Their images in the regular cohomology of the flag variety are a linear basis.*
2. $\int_{X_w^{\tau w_0}} \mathfrak{F}_v^\tau = \delta_{wv}$, where $\int_{X_w^{\tau w_0}}$ is defined below.

THEOREM 2.6 (Arabia). *A set of equivariant cohomology classes with properties (1) and (2) as in Theorem 2.5 provides a linear basis for $H_T^*(G/T)$ as a module over H_T^* .*

Following Arabia, the integral $\int_{X_w^{id}}$ is defined by integrating over a certain (choice of) smooth variety which maps birationally to X_w^{id} .

THEOREM 2.7 (Bott–Samelson). *There is a (non-canonical) tower of $\mathbb{C}P^1$ bundles over $\mathbb{C}P^1$, denoted BS_{w_0} , which maps birationally to G/B such that, for every $w \in W$, there is a smooth subvariety BS_w of BS_{w_0} which maps birationally to $X_w^{id} \subset G/B$.*

While these *Bott–Samelson resolutions* are not canonical, we always have that

$$\begin{array}{ccc} BS_w & \xrightarrow{i_{BS_w}} & BS_{w_0} \\ \pi_w \downarrow & & \downarrow \pi_{w_0} \\ X_w & \xrightarrow{i_w} & G/T \end{array}$$

is a T -equivariant commutative diagram (see [5]), where π_w is generically one-to-one. In the induced commutative diagram

$$\begin{array}{ccc} H_T^*(BS_w) & \xleftarrow{i_{BS_w}^*} & H_T^*(BS_{w_0}) \\ \pi_w^* \uparrow & & \uparrow \pi_{w_0}^* \\ H_T^*(X_w) & \xleftarrow{i_w^*} & H_T^*(G/T) \end{array}$$

the map π_w^* is an injection for all w . Thus for any $\alpha \in H_T^*(G/T)$,

$$\pi_w^* i_w^* \alpha = i_{BS_w}^* \pi_{w_0}^* \alpha.$$

As each BS_w is a smooth submanifold of BS_{w_0} , we define

$$\int_{X_w^{id}} i_w^* \alpha := (\pi_w)_* \int_{BS_w} i_{BS_w}^* \pi_{w_0}^* \alpha$$

where $(\pi_w)_*$ is the pushforward induced by π_w . This whole construction can equally well be done for the varieties X_w^τ to define $\int_{X_w^\tau}$ for all τ, w .

Proof of Theorem 2.5. That the classes \mathfrak{T}_w^τ restrict to generators of the regular cohomology of the flag manifolds (1) is by construction. The \mathfrak{T}_w^τ restrict to Schubert polynomials (permuted by τ), shown in [2] to be such generators.

The integrality condition (2) is also easy to show, using the Atiyah–Bott/Berline–Vergne fixed point theorem. Using Arabia’s definition of the integral, the fixed point theorem says that

$$\int_{X_w^{\tau w_0}} \mathfrak{F}_w^\tau = \sum_{p \in (BS_w^{\tau w_0})^T} \frac{(\pi_{w_0}^* \mathfrak{F}_w^\tau)|_p}{e_{BS_w^{\tau w_0}}(p)},$$

where $e_{BS_w^{\tau w_0}}(p)$ is the equivariant Euler class of $p \in BS_w^{\tau w_0}$. The set of fixed points for the T action on $X_w^{\tau w_0}$ is

$$\begin{aligned} \{z: z \leq_{\tau w_0} w\} &= \{z: w_0 \tau^{-1} z \leq w_0 \tau^{-1} w\} \\ &= \{z: \tau^{-1} z \geq \tau^{-1} w\} \\ &= \{z: z \geq_\tau w\} \end{aligned}$$

while $\text{supp } \mathfrak{F}_v^\tau = \{z: z \leq_\tau v\}$.

If $v = w$, there is only one point contributing to the integral, i.e.

$$\int_{X_w^{\tau w_0}} \mathfrak{F}_w^\tau = \frac{\mathfrak{F}_w^\tau|_w}{e(w)},$$

where $e(w)$ is the equivariant Euler class of $w \in BS_w^{\tau w_0}$ ($BS_w^{\tau w_0}$ has exactly one point in the fibre over $w \in G/T$). A quick computation shows that the denominator of this expression is the product of the roots pointing into the variety $X_w^{\tau w_0}$, while the numerator is the product of the roots pointing out of X_w^τ . As these are equivalent, the quotient is 1.

If $v \neq w$, there are two possibilities. If $l(\tau^{-1}v) \leq l(\tau^{-1}w)$, then the integral is clearly 0 because the sets $(X_w^{\tau w_0})^T$ and $\text{supp } \mathfrak{F}_w^\tau$ do not intersect. If $l(\tau^{-1}v) > l(\tau^{-1}w)$, there may be a contribution to the integral by points $z \in W$ such that $v \geq_\tau z \geq_\tau w$. However, $\deg \mathfrak{F}_v^\tau = \deg \mathfrak{F}_{\tau^{-1}v}^{id} = n(n-1) - l(\tau^{-1}v)$ and

$$\begin{aligned} \dim X_w^{\tau w_0} &= n(n-1) - \deg \mathfrak{F}_w^{\tau w_0} = n(n-1) - [n(n-1) - l(w_0 \tau^{-1}w)] \\ &= n(n-1) - l(\tau^{-1}w). \end{aligned}$$

Then $l(\tau^{-1}v) > l(\tau^{-1}w)$ implies $\dim X_w^{\tau w_0} > \deg \mathfrak{F}_v^\tau$, which implies that each contributing term in the integral has a denominator of higher degree than the numerator. Because the integral will be polynomial, these terms must sum to zero. ■

3. THE SYMPLECTIC PICTURE OF $H_T^*(\mathcal{O}_\lambda)$

Recall from Theorem 1.5 that $M_\xi^\mu \subset M$ is the set of points whose image under the moment map lies to one side of the hyperplane ξ_μ^\perp through μ in \mathfrak{t}^* , i.e.

$$M_\xi := \{m \in M \mid \langle \phi(m), \xi \rangle \leq \langle \mu, \xi \rangle\}.$$

THEOREM 3.1. *Let \mathcal{O}_λ be a generic $SU(n)$ coadjoint orbit through $\lambda \in \mathfrak{t}^*$, and let $\alpha \in H_T^*(\mathcal{O}_\lambda)$ be an equivariant cohomology class with $\text{supp } \alpha \subset (\mathcal{O}_\lambda)_\xi^\mu$. Then there exists some $\tau \in W$ such that*

$$\alpha = \sum_{v \in W} a_v^\tau \mathfrak{T}_v^\tau,$$

where $a_v^\tau \in H_T^*$ non-zero implies $\text{supp } \mathfrak{T}_v^\tau \subset (\mathcal{O}_\lambda)_\xi^\mu$. Furthermore, τ may be chosen as any element of the Weyl group such that ξ realizes its minimum at $\phi(\lambda_\tau)$.

Note that this theorem is a symplectic, rather than topological statement, i.e. it depends on the choice of λ and μ . We first need a lemma about the behavior of the ξ with regard to the Bruhat order. We let e_i be coordinate functions on \mathfrak{t}^* , so that if $\lambda \in \mathfrak{t}^*$ is written $(\lambda_1, \dots, \lambda_n)$, then $e_i(\lambda) = \lambda_i$. Recall that we have chosen λ such that $\lambda_1 > \dots > \lambda_n$. Let $\xi_\mu^\perp \subset \mathfrak{t}^*$ indicate a hyperplane perpendicular to $\xi \in \mathfrak{t}$ through μ .

LEMMA 3.1. *Let $\xi \in \mathfrak{t}$ such that among points λ_w , ξ attains its minimum at $w = id$. Then ξ respects the Bruhat order, i.e. $\xi(\lambda_v) \leq \xi(\lambda_w)$ if $v \leq w$ in the Bruhat order.*

Proof. Write $\xi = \sum_{i=1}^n b_i e_i$ in the basis given above, with $b_i \in \mathbb{R}$. Comparing $\xi(\lambda_i d)$ with $\xi(\lambda_{s_i})$, we have (by our minimality assumption) $b_i \lambda_i + b_{i+1} \lambda_{i+1} \leq b_i \lambda_{i+1} + b_{i+1} \lambda_i$. Since $\lambda_i > \lambda_{i+1}$, we find $b_i \leq b_{i+1}$. Over all i we find have

$$b_1 \leq \dots \leq b_n. \tag{3}$$

If $v < w$, then there is a sequence of length decreasing simple reflections $s_{i_1} \dots s_{i_k}$ such that $v = s_{i_1} \dots s_{i_k} w$. For each reflection, we claim the value of ξ decreases. Suppose $s_{i_k} = s_1$. Then

$$\xi(\lambda_{s_1 w}) = \sum_i b_i \lambda_{w^{-1} s_1(i)} = b_1 \lambda_{w^{-1}(2)} + b_2 \lambda_{w^{-1}(1)} + \sum_{i=3} b_i \lambda_{w^{-1}(i)}$$

and the difference $\xi(\lambda_w) - \xi(\lambda_{s_1 w})$ is

$$\begin{aligned} & b_1 \lambda_{w^{-1}(1)} + b_2 \lambda_{w^{-1}(2)} - (b_1 \lambda_{w^{-1}(2)} + b_2 \lambda_{w^{-1}(1)}) \\ &= b_2(\lambda_{w^{-1}(2)} - \lambda_{w^{-1}(1)}) - b_1(\lambda_{w^{-1}(2)} - \lambda_{w^{-1}(1)}). \end{aligned}$$

But $s_1 w < w$ if and only if $w^{-1}s_1 < w^{-1}$, which implies that $w^{-1}(2) < w^{-1}(1)$, and thus $\lambda_{w^{-1}(2)} > \lambda_{w^{-1}(1)}$. As the difference is positive, by (3) we have

$$b_1(\lambda_{w^{-1}(2)} - \lambda_{w^{-1}(1)}) \leq b_2(\lambda_{w^{-1}(2)} - \lambda_{w^{-1}(1)})$$

and therefore $\xi(\lambda_w) - \xi(\lambda_{s_1 w}) \geq 0$, as desired. The same proof applies for $s_{i_k} = s_j$ for any length-reducing simple transposition s_j . Continuing inductively we obtain the result.

This lemma generalizes quite easily to the case where the linear functional is minimized at λ_τ for any $\tau \in W$.

LEMMA 3.2. *Let ξ_τ be a linear function on \mathfrak{t}^* which attains its minimum on λ_τ . Then $v \leq_\tau w$ implies $\xi_\tau(\lambda_v) \leq \xi_\tau(\lambda_w)$.*

Proof. By definition, $v \leq_\tau w$ if and only if $\tau^{-1}v \leq \tau^{-1}w$ in the Bruhat order. Then for any ξ which is minimized at λ_{id} , we have $\xi(\lambda_{\tau^{-1}v}) \leq \xi(\lambda_{\tau^{-1}w})$ by Lemma 3.1. Define $\xi(\lambda_w) := \xi_\tau(\lambda_{\tau w})$ for all $w \in W$. Then ξ_τ minimal at λ_τ implies ξ minimal at λ_{id} , which then implies

$$\xi_\tau(\lambda_v) = \xi(\lambda_{\tau^{-1}v}) \leq \xi(\lambda_{\tau^{-1}w}) = \xi_\tau(\lambda_w).$$

Proof of Theorem 3.1. Let τ be such that $\xi(\lambda_\tau)$ is minimal. For any $\alpha \in H_T^*(M)$ we can write

$$\alpha = \sum_{v \in W} a_v^\tau \mathfrak{I}_v^\tau$$

with $a_v^\tau \in H_T^*$. We assume that $\text{supp } \alpha$ is contained in M_ξ^μ and show that $\text{supp } \mathfrak{I}_v^\tau$ must also be contained in M_ξ^μ whenever $a_v^\tau \neq 0$.

Suppose not. Let $F = \{q \in W \text{ such that } \alpha|_q = 0 \text{ but not all } a_v^\tau \mathfrak{I}_v^\tau|_q = 0\}$. F is not empty by assumption. Choose any $q \in F$ such that there are no points $q' \in F$ with $q' >_\tau q$. If $a_q^\tau \mathfrak{I}_q^\tau = 0$, then since $q \in F$, there exists some q' such that $a_{q'}^\tau \mathfrak{I}_{q'}^\tau|_q \neq 0$. Furthermore, $\text{supp } \mathfrak{I}_{q'}^\tau = (X_{q'}^\tau)^T$ implies $q \in (X_{q'}^\tau)^T$ and thus $q <_\tau q'$. Then by Lemma 3.2, $\xi_\tau(q) \leq \xi_\tau(q')$, which implies that $q' \in F$

since $\text{supp } \alpha \subset M_\xi^\mu$. Then q was not a maximal element (in the $>_\tau$ ordering) of F . We now have that $\alpha|_q = \sum_{v \in W} a_v^\tau \mathfrak{I}_v^\tau|_q = 0$ implies

$$a_q^\tau \mathfrak{I}_q^\tau = - \sum_{v \neq q} a_v^\tau \mathfrak{I}_v^\tau|_q \neq 0.$$

By the same reasoning, for $v >_\tau q$, we have $a_v^\tau = 0$ (otherwise $v \in F$ and by Lemma 3.2, q would not be maximal in the $>_\tau$ order). This implies

$$\sum_{v \not\leq_\tau q} a_v^\tau \mathfrak{I}_v^\tau|_q \neq 0.$$

But $\text{supp } \mathfrak{I}_v^\tau = (X_v^\tau)^T$ implies $q \in X_v^\tau$ for some v for this sum to be non-zero, which in turn implies $q \leq_\tau v$, a contradiction. ■

4. PROOF OF THEOREMS 1.1 AND 1.2

First consider the case where λ is generic (Theorem 1.1). We show that the kernel of the map

$$\kappa_\mu: H_T^*(\mathcal{O}_\lambda) \rightarrow H^*(\mathcal{O}_\lambda//T(\mu))$$

is generated by the set of $\partial_v(x, u_\tau)$ listed in Theorem 1.1. The theorem then follows by the quotient relation

$$H^*(\mathcal{O}_\lambda//T(\mu)) = H_T^*(\mathcal{O}_\lambda)/\ker \kappa_\mu$$

and the direct computation of $H_T^*(\mathcal{O}_\lambda)$ in Section 2.1.

We go about this by using support considerations. Consider the class \mathfrak{I}_w^τ with support $(X_w^\tau) = \{v \in W : v \leq_\tau w\}$. The functions

$$\eta_k^\tau = \sum_{i=k+1}^n e_{\tau(i)}$$

obtain their maxima at $v = \lambda_w$ and their minima at $v = \lambda_\tau$. Thus $\eta_k^\tau(\lambda_w) = \sum_{i=k+1}^n e_{\tau(i)}(\lambda_w) < \sum_{i=k+1}^n \mu_{\tau(i)}$ implies $\sum_{i=k+1}^n e_{\tau(i)}(\lambda_w) < \sum_{i=k+1}^n \mu_{\tau(i)}$ by Lemma 3.2. We calculate

$$\begin{aligned} \sum_{i=k+1}^n e_{\tau(i)}(\lambda_w) &= \sum_{i=k+1}^n e_{\tau(i)}((\lambda_{w^{-1}(1)}, \dots, \lambda_{w^{-1}(n)})) \\ &= \sum_{i=k+1}^n \lambda_{w^{-1}\tau(i)}. \end{aligned}$$

If $\sum_{i=k+1}^n \lambda_{w^{-1\tau(i)}} < \sum_{i=k+1}^n \mu_{\tau(i)}$, then $\text{supp } \mathfrak{I}_w^\tau = (X_w^\tau)^T = \{v: v \leq_\tau w\}$ lies in $M_{\eta_k}^\mu$, which by Theorem 1.2 proves that \mathfrak{I}_w^τ is in $\ker \kappa_\mu$. As $\mathfrak{I}_w^\tau(x, u) = \partial_{w^{-1\tau}} \Delta(x, u_\tau)$, we have that $\partial_v \Delta(x, u_\tau) \in \ker \kappa_\mu$ if $\sum_{i=k+1}^n \lambda_{v(i)} < \sum_{i=k+1}^n \mu_{\tau(i)}$, as stated in Theorem 1.1.

We need to show that these \mathfrak{I}_w^τ generate the kernel. Let $\alpha \in H_T^*(M)$ be a homogeneous class such that $\text{supp } \alpha \subset M_\xi^\mu$. By Theorem 3.1 we can write

$$\alpha = \sum_{w \in W} a_w^\tau \mathfrak{I}_w^\tau$$

where $\text{supp } \mathfrak{I}_w^\tau \subset (X_w^\tau)^T = \{v: v \leq_\tau w\}$.

We show $\text{supp } \mathfrak{I}_w^\tau \subset M_{\eta_k}^\mu$ for some k . Again by Lemma 3.2, if $v \leq_\tau w$, we have $\eta_k^\tau(\lambda_v) \leq \eta_k^\tau(\lambda_w)$ for all k . It is thus equivalent to show that

$$\eta_k^\tau(\lambda_w) < \eta_k^\tau(\mu) \tag{4}$$

for some k .

Suppose that the equality (4) does not hold for any k . We have a series of inequalities

$$\begin{aligned} \lambda_{w^{-1\tau(n)}} &\geq \mu_{\tau(n)} \\ \lambda_{w^{-1\tau(n-1)}} + \lambda_{w^{-1\tau(n)}} &\geq \mu_{\tau(n-1)} + \mu_{\tau(n)} \\ &\vdots \\ \lambda_{w^{-1\tau(2)}} + \dots + \lambda_{w^{-1\tau(n)}} &\geq \mu_{\tau(2)} + \dots + \mu_{\tau(n)}. \end{aligned}$$

Note that $\text{supp } \alpha \subset M_\xi^\mu$ is equivalent to

$$\lambda_w \in \text{supp } \alpha \quad \text{implies} \quad \xi(\lambda_w) < \xi(\mu).$$

For $\xi = \sum_{i=1}^n b_i e_i$ by the same argument as that used in the proof of Lemma 3.1 we find that $b_{\tau(1)} \leq \dots \leq b_{\tau(n)}$. Therefore,

$$\begin{aligned} (b_{\tau(n)} - b_{\tau(n-1)}) \lambda_{w^{-1\tau(n)}} &\geq (b_{\tau(n)} - b_{\tau(n-1)}) \mu_{\tau(n)} \\ (b_{\tau(n-1)} - b_{\tau(n-2)}) (\lambda_{w^{-1\tau(n-1)}} + \lambda_{w^{-1\tau(n)}}) &\geq (b_{\tau(n-1)} - b_{\tau(n-2)}) (\mu_{\tau(n-1)} + \mu_{\tau(n)}) \\ &\vdots \\ (b_{\tau(2)} - b_{\tau(1)}) (\lambda_{w^{-1\tau(2)}} + \dots + \lambda_{w^{-1\tau(n)}}) &\geq (b_{\tau(2)} - b_{\tau(1)}) (\mu_{\tau(2)} + \dots + \mu_{\tau(n)}). \end{aligned}$$

Summing the inequalities and using $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i = 0$ we obtain

$$b_{\tau(n)} \lambda_{w^{-1\tau(n)}} + \dots + b_{\tau(1)} \lambda_{w^{-1\tau(1)}} \geq b_{\tau(n)} \mu_{\tau(n)} + \dots + b_{\tau(1)} \mu_{\tau(1)},$$

which is of course equivalent to

$$b_n \lambda_{w^{-1}(n)} + \cdots + b_1 \lambda_{w^{-1}(1)} \geq b_n \mu_n + \cdots + b_1 \mu_1,$$

or $\xi(\lambda_w) \geq \xi(\mu)$, a contradiction.

It follows as an immediate corollary that the Tolman–Weitsman Theorem (1.5) can be refined in the following way. Let $J \subset \mathfrak{t}$ be the set of fundamental weights, permuted by the Weyl group.

COROLLARY 4.1. *Let $M = \mathcal{O}_\lambda$ be the coadjoint orbit of $SU(n)$ through a generic choice of $\lambda \in \mathfrak{t}^*$. Let M_ξ^μ and K_ξ be as in Theorem 1.5. The kernel of the map*

$$\kappa_\mu: H_T^*(M) \rightarrow H_T^*(M//T(\mu))$$

is the ideal generated by $K = \bigcup_{\xi \in J} K_\xi$. Equivalently, a sufficient set of hyperplanes in Theorem 1.5 is the set of all hyperplanes through $\mu \in \mathfrak{t}^$ which are parallel to the codimension-one walls of the moment polytope.*

Proof. The fundamental weights of $SU(n)$ are η_k^τ . We show how these elements are perpendicular to hyperplanes parallel to codimension-one walls in the image of the moment map. For the moment map $\phi: M \rightarrow \mathfrak{t}^*$, codimension-one walls of the moment polytope consist of the image in \mathfrak{t}^* of the set of points fixed by some $S^1 \subset T$ and having an effective T/S^1 action.

The fixed point set of a codimension-one wall for \mathcal{O}_λ is the permutations of a partition of n letters into two sets, of cardinality k and $n - k$, respectively. One easily sees that $\eta_k^\tau = \sum_{i=k+1}^n e_{\tau(i)}$ is constant on the partition $(\{\tau(1), \dots, \tau(k)\}, \{\tau(k+1), \dots, \tau(n)\})$ and its permutations by $S_k \times S_{n-k}$. Thus $(\eta_k^\tau)^\perp$ is parallel to this wall. ■

We proceed to prove Theorem 1.2. We prove a little lemma which shows that, as a subring of $H_T^*(Fl(\mathbb{C}^n))$, the cohomology $H_T^*(Gr(k, n))$ is linearly generated by classes \mathfrak{Z}_w^τ that are symmetric in certain variables.

LEMMA 4.1. *Let $\alpha \in H_T^*(Gr(k, n))$. Then α can be written*

$$\alpha = \sum_{w \in W} a_w^\tau \mathfrak{Z}_w^\tau$$

with \mathfrak{Z}_w^τ symmetric in $(x_{\tau^{-1}(1)}, \dots, x_{\tau^{-1}(k)})$ and in $(x_{\tau^{-1}(k+1)}, \dots, x_{\tau^{-1}(n)})$.

Proof. Let $i: Fl(\mathbb{C}^n) \rightarrow Gr(k, n)$ be the forgetful map and $i^*: H_T^*(Gr(k, n)) \rightarrow H_T^*(Fl(\mathbb{C}^n))$ be the induced inclusion in equivariant cohomology. The map $Fl(\mathbb{C}^n)^T \rightarrow Gr(k, n)^T$ under an identification $Fl(\mathbb{C}^n)^T \cong W$ sends $S_n \rightarrow S_n/(S_k \times S_{n-k})$. Any $\alpha \in H_T^*(Fl(\mathbb{C}^n))$ which is the image of i^* is therefore

constant in its restriction to a fixed point $p \in Fl(\mathbb{C}^n)^T$ and its permutations by $S_k \times S_{n-k}$. By Section 2.4 we may write

$$\alpha = \sum_{w \in W} a_w^\tau \mathfrak{I}_w^\tau.$$

We need to show that \mathfrak{I}_w^τ are symmetric in the relevant variables, which is equivalent to being in the image of i^* . We show this first for $\tau = id$.

Suppose not all \mathfrak{I}_w^{id} were symmetric in x_1, \dots, x_k and in x_{k+1}, \dots, x_n . Let $v \in W$ be the longest length element such that $a_v^{id} \neq 0$ and \mathfrak{I}_v^{id} not symmetric in this sense. Then v must be the longest element in W in the orbit of $S_k \times S_{n-k} \cdot v$. If not, then since α must be equal on all points in $S_k \times S_{n-k} \cdot v$, there must be some \mathfrak{I}_w^{id} with $a_w^{id} \neq 0$, $\mathfrak{I}_v^{id} \neq \mathfrak{I}_w^{id}$ and $l(w) > l(v)$. But \mathfrak{I}_w^{id} symmetric (as v is the longest element with \mathfrak{I}_v^{id} not symmetric) implies α not symmetric.

For v the longest element in the orbit, however, \mathfrak{I}_v^{id} is symmetric. For any $s_i \in S_k \times S_{n-k}$, $s_i v < v$ implies $\partial_i \mathfrak{I}_v^{id} = 0$, or \mathfrak{I}_v^{id} is symmetric in x_i and x_{i+1} , $i = 1, \dots, k-1, k+1, \dots, n-1$.

Similarly, one proves for every τ that $\alpha = \sum_{w \in W} a_w^\tau \mathfrak{I}_w^\tau$ implies that each contributing term \mathfrak{I}_w^τ is symmetric in $(x_{\tau^{-1}(1)}, \dots, x_{\tau^{-1}(k)})$ and in $(x_{\tau^{-1}(k+1)}, \dots, x_{\tau^{-1}(n)})$. ■

It now follows that for $\alpha \in H_T^*(Gr(k, n)_v)$, Theorem 3.1 holds with all \mathfrak{I}_v^τ symmetric in the appropriate variables. Theorem 1.2 then follows by the same proof as that for Theorem 1.1.

5. EXAMPLES

We present two examples for $n = 4$, one of a generic coadjoint orbit with two positive and two negative eigenvalues, and the second of a degenerate coadjoint orbit which is the 2-Grassmannian in \mathbb{C}^4 .

5.1. $SU(4)$ Generic Coadjoint Orbits

It has been shown that for $SU(n)$ coadjoint orbits that the reduction at 0 of orbits whose isospectral sets $\langle \lambda_1, \dots, \lambda_n \rangle$ are close to $\langle a, \dots, a, b \rangle$ or $\langle a, b, \dots, b \rangle$ for $a > b$ are again coadjoint orbits, now of $SU(n-1)$ [13]. For 0-weight varieties, this means that if $n-1$ eigenvalues are above (or below) zero, the associated 0-weight variety will be a coadjoint orbit of $SU(n-1)$. Here we use the methods developed above to compute the cohomology ring of the 0-weight variety of $SU(4)$ in the case where $\lambda_1 > \lambda_2 > 0 > \lambda_3 > \lambda_4$.

There is one choice that has an effect on the kernel of the Kirwan map $\kappa: H_T^*(\mathcal{O}_\lambda) \rightarrow H^*(\mathcal{O}_\lambda//T(0))$: either $\lambda_2 + \lambda_3 > 0$ (which implies $\lambda_1 + \lambda_4 < 0$),

or $\lambda_2 + \lambda_3 < 0$ (which implies $\lambda_1 + \lambda_4 > 0$). In this case, the resulting cohomology rings are isomorphic, and so we choose $\lambda_2 + \lambda_3 > 0$. This forces the ordering on the partial sums:

$$\lambda_1 + \lambda_2 > \lambda_1 + \lambda_3 > \lambda_2 + \lambda_3 > 0 > \lambda_1 + \lambda_4 > \lambda_2 + \lambda_4 > \lambda_3 + \lambda_4. \quad (5)$$

THEOREM 5.1. *For \mathcal{O}_λ the coadjoint orbit through λ satisfying the relation (5), the cohomology of the zero weight variety is described by*

$$H^*(\mathcal{O}_\lambda//T(0)) = \frac{\mathbb{C}[x_1, \dots, x_4, u_1, \dots, u_4]}{(\prod (1 + u_i) - \prod (1 + x_i), \sum_i u_i, \alpha_1, \dots, \alpha_{14})},$$

where α_i for $i = 1, \dots, 14$ are the following degree 4 classes:

$$\begin{aligned} \alpha_1 &= (x_1 - u_4)(x_2 - u_4) & \alpha_3 &= (x_1 - u_2)(x_2 - u_2) \\ \alpha_2 &= (x_1 - u_3)(x_2 - u_3) & \alpha_4 &= (x_1 - u_1)(x_2 - u_1) \\ \alpha_5 &= x_1^2 + x_1x_2 + x_2^2 - (u_2 + u_3 + u_4)(x_1 + x_2) + (u_2u_3 + u_2u_4 + u_3u_4) \\ \alpha_6 &= x_1^2 + x_1x_2 + x_2^2 - (u_1 + u_3 + u_4)(x_1 + x_2) + (u_1u_3 + u_1u_4 + u_3u_4) \\ \alpha_7 &= x_1^2 + x_1x_2 + x_2^2 - (u_1 + u_2 + u_4)(x_1 + x_2) + (u_1u_2 + u_1u_4 + u_2u_4) \\ \alpha_8 &= x_1^2 + x_1x_2 + x_2^2 - (u_1 + u_2 + u_3)(x_1 + x_2) + (u_1u_2 + u_2u_3 + u_1u_3) \\ \alpha_9 &= x_1x_2 + x_2x_3 + x_1x_3 - (u_3 + u_4)(x_1 + x_2 + x_3) + (u_3^2 + u_3u_4 + u_4^2) \\ \alpha_{10} &= x_1x_2 + x_2x_3 + x_1x_3 - (u_2 + u_4)(x_1 + x_2 + x_3) + (u_2^2 + u_2u_4 + u_4^2) \\ \alpha_{11} &= x_1x_2 + x_2x_3 + x_1x_3 - (u_1 + u_4)(x_1 + x_2 + x_3) + (u_1^2 + u_1u_4 + u_4^2) \\ \alpha_{12} &= x_1x_2 + x_2x_3 + x_1x_3 - (u_2 + u_3)(x_1 + x_2 + x_3) + (u_2^2 + u_2u_3 + u_3^2) \\ \alpha_{13} &= x_1x_2 + x_2x_3 + x_1x_3 - (u_1 + u_3)(x_1 + x_2 + x_3) + (u_1^2 + u_1u_3 + u_3^2) \\ \alpha_{14} &= x_1x_2 + x_2x_3 + x_1x_3 - (u_1 + u_2)(x_1 + x_2 + x_3) + (u_1^2 + u_1u_2 + u_2^2). \end{aligned}$$

COROLLARY 5.1. *Let \mathcal{O}_λ be a non-extremal coadjoint orbit of $SU(4)$. The Poincaré polynomial for $\mathcal{O}_\lambda//T(0)$ is $1 + 6t^2 + 6t^4 + t^6$.*

For this very symmetric case, one can check the Betti numbers using the Morse theory methods of Kirwan; see [15], where she does similar examples. For the Betti numbers of this example, it is easier to use Kirwan's methods, because the indices of critical points are easy to calculate using the symmetry. When the entries in λ are distributed randomly, however, it may be quite difficult to read off the index of a particular critical point. The method proposed here, while requiring many more tedious calculations, has the advantage of being straightforward.

Proof of Theorem. First consider $\tau = id$. The fundamental weights which are minimized at λ_{id} (among λ_v) are $\xi = e_4$, $\xi = e_3 + e_4$ and $\xi = e_2 + e_3 + e_4$. The set of points λ_w such that $\xi(\lambda_w) < \xi(0)$ for one of these choices of ξ are as follows, listed by length:

$$\text{For } l(w) = 0, \quad w = [1234]$$

$$\text{For } l(w) = 1, \quad w = [2134], [1243], [1324]$$

$$\text{For } l(w) = 2, \quad w = [2143], [1423], [3124], [2314], [1342]$$

$$\text{For } l(w) = 3, \quad w = [3214], [3142], [2341], [2413], [4123]$$

$$\text{For } l(w) = 4, \quad w = [3241], [4132], [4213].$$

Correspondingly, we need to find the set of all $\mathfrak{T}_w^\tau = \mathfrak{T}_w^{id}$ for these w . The smallest degree \mathfrak{T}_w^{id} are those for which $l(w) = 4$. We compute these classes:

$$\text{For } w = [3241] = s_1 s_2 s_1 s_3,$$

$$\begin{aligned} \mathfrak{T}_w^{id} &= \partial_{w^{-1}} \Delta = \partial_3 \partial_1 \partial_2 \partial_1 (x_1 - u_2)(x_1 - u_3) \\ &\quad \times (x_1 - u_4)(x_2 - u_3)(x_2 - u_4)(x_3 - u_4) \\ &= \partial_3 \partial_1 \partial_2 (x_1 - u_3)(x_1 - u_4)(x_2 - u_3)(x_2 - u_4)(x_3 - u_4) \\ &= \partial_3 \partial_1 (x_1 - u_3)(x_1 - u_4)(x_2 - u_4)(x_3 - u_4) \\ &= \partial_3 (x_1 - u_4)(x_2 - u_4)(x_3 - u_4) \\ &= (x_1 - u_4)(x_2 - u_4). \end{aligned}$$

$$\text{For } w = [4132] = s_3 s_2 s_3 s_1,$$

$$\begin{aligned} \mathfrak{T}_w^{id} &= \partial_{w^{-1}} \Delta = \partial_1 \partial_3 \partial_2 \partial_3 (x_1 - u_2)(x_1 - u_3) \\ &\quad \times (x_1 - u_4)(x_2 - u_3)(x_2 - u_4)(x_3 - u_4) \\ &= \partial_1 \partial_3 \partial_2 (x_1 - u_2)(x_1 - u_3)(x_1 - u_4)(x_2 - u_3)(x_2 - u_4) \\ &= \partial_1 \partial_3 (x_1 - u_2)(x_1 - u_3)(x_1 - u_4)(x_2 + x_3 - (u_3 + u_4)) \\ &= \partial_1 (x_1 - u_2)(x_1 - u_3)(x_1 - u_4) \\ &= x_1^2 + x_1 x_2 + x_2^2 - (u_2 + u_3 + u_4)(x_1 + x_2) \\ &\quad + (u_2 u_3 + u_2 u_4 + u_3 u_4). \end{aligned}$$

For $w = [4213] = s_3 s_1 s_2 s_1$,

$$\begin{aligned}
\mathfrak{T}_w^{id} &= \partial_{w^{-1}A} = \partial_1 \partial_2 \partial_1 \partial_3 (x_1 - u_2)(x_1 - u_3) \\
&\quad \times (x_1 - u_4)(x_2 - u_3)(x_2 - u_4)(x_3 - u_4) \\
&= \partial_1 \partial_2 \partial_1 (x_1 - u_2)(x_1 - u_3)(x_1 - u_4)(x_2 - u_3)(x_2 - u_4) \\
&= \partial_1 \partial_2 (x_1 - u_3)(x_1 - u_4)(x_2 - u_3)(x_2 - u_4) \\
&= \partial_1 (x_1 - u_3)(x_1 - u_4)(x_2 + x_3 - (u_3 + u_4)) \\
&= x_1 x_2 + x_1 x_3 + x_2 x_3 - (u_3 + u_4)(x_1 + x_2 + x_3) \\
&\quad + (u_3^2 + u_3 u_4 + u_4^2).
\end{aligned}$$

These three degree 4 cohomology classes can be permuted by the Weyl group to obtain other classes in the kernel, for if ξ is minimized at λ_{id} , $\xi = \sum_{i=1}^n b_i e_i$, then $\xi_\tau = \sum_{i=1}^n b_i e_{\tau(i)}$ is minimized at λ_τ . If $\xi(w) < \xi(0) = 0$, then $\xi_\tau(\lambda_{\tau w}) = \xi(\lambda_w) < \xi(0) = \xi_\tau(0) = 0$ implies that $\mathfrak{T}_{\tau w}^\tau \in \ker \kappa$. But $\mathfrak{T}_{\tau w}^\tau := \mathfrak{T}_w^{id}(x, u_\tau)$, or permutations in the u -variables of the three classes we just computed. The permutations of \mathfrak{T}_w^{id} for $w = [3241]$ are the classes $\alpha_1, \dots, \alpha_4$ listed in Theorem 5.1. The permutations of \mathfrak{T}_w^{id} for $w = [4132]$ are the classes $\alpha_5, \dots, \alpha_8$, and those for $w = [4213]$ are the classes $\alpha_9, \dots, \alpha_{14}$ (see Fig. 1).

These classes are independent contributions to the kernel of

$$\kappa: H_{\mathbb{T}}^*(\mathcal{O}_\lambda) \rightarrow H^*(\mathcal{O}_\lambda//T(0)),$$

as can be verified by a laborious computation. While these classes are the only degree 4 classes, we theoretically must calculate the remaining (higher) degree classes in the kernel. However, a direct computation shows that the Poincaré polynomial of $H_{\mathbb{T}}^*(\mathcal{O}_\lambda)/\langle \alpha_1, \dots, \alpha_{14} \rangle$ is $1 + 6t^2 + 6t^4 + t^6$, and since the $\dim_{\mathbb{R}} \mathcal{O}_\lambda//T(0) = 6$, there cannot be any further contributions to the kernel. ■

5.2. The Grassmannian $Gr(2, 4)$

The Grassmannian case is very similar in computation to the case of the generic coadjoint orbit of $SU(4)$ with the eigenvalues ordered above. As a (degenerate) coadjoint orbit, $Gr(2, 4)_v$ is the orbit through a point $v \in \mathfrak{t}^*$ where the orbit has two distinct eigenvalues instead of four. We label them $v = (v_1, v_1, v_2, v_2)$, so that $v_1 + v_1 + v_2 + v_2 = 0$, or $v_1 = -v_2$. This case is in some sense a limit case of the example above, as λ_1 approaches λ_2 , and λ_3 approaches λ_4 in Expression (5).

We compute the cohomology of the symplectic reduction $Gr(2, 4)_v//T(\mu)$ where the point of reduction $\mu \in \mathfrak{t}^*$ cannot be zero, as zero is not a regular value of the moment map in this case. Choose $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ with

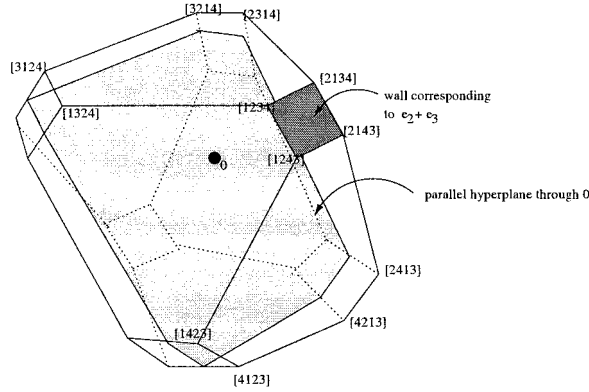


FIG. 1. The hyperplane in light gray indicates a degree 4 class in the kernel of the Kirwan map $\kappa: H_T^*(\mathcal{O}_\lambda) \rightarrow H^*(\mathcal{O}_\lambda//T(0))$ whose support lies on the labeled points to one side of the hyperplane.

$\sum_{i=1}^4 \mu_i = 0$ and $\mu_1 > \mu_2 > \mu_3 > 0$ but close enough to zero that μ is in the image of the moment polytope.

THEOREM 5.2. For \mathcal{O}_ν the coadjoint orbit through ν satisfying the relation above, the cohomology of the μ weight variety for μ as above is described by

$$H^*(Gr(2, 4)_\nu//T(\mu)) = \frac{\mathbb{C}[x_1 + x_2, x_3 + x_4, x_1x_2, x_3x_4, u_1, \dots, u_4]}{(\prod (1 + u_i) - \prod (1 + x_i), \sum_i u_i, \beta_1, \beta_2, \beta_3, \alpha_1, \dots, \alpha_8)},$$

where the α_i are the degree 4 classes listed in Theorem 5.1 and β_i are the following degree 2 classes:

$$\begin{aligned} \beta_1 &= (x_1 + x_2 - (u_1 + u_2)) \\ \beta_2 &= (x_1 + x_2 - (u_1 + u_4)) \\ \beta_3 &= (x_1 + x_2 - (u_1 + u_3)). \end{aligned}$$

Proof. It should first be noted that indeed there is repetition in the classes α_i and β_i . This is done however to emphasize that the classes in the kernel include the classes in the kernel for the generic case which are in the image of the map $H_T^*(Gr(2, 4)) \hookrightarrow H_T^*(SU(4)/T)$ induced by the forgetful map from the flag variety to the Grassmannian.

We do a similar computation as above to find classes in the kernel of the map

$$\kappa_\mu: H_T^*(Gr(2, 4)_\nu) \rightarrow H^*(Gr(2, 4)_\nu//T(\mu)). \tag{6}$$

The eight classes $\alpha_1, \dots, \alpha_8$ are easily seen to be in the kernel of κ_μ . Theorem 1.1 states that if $\xi_\tau^j(\lambda_w) < \xi_\tau^j(0)$, then $\mathfrak{T}_w^\tau \in \kappa_0$ for the case of a generic coadjoint orbit reduced at 0. But for μ small enough, $\xi_\tau^j(\lambda_w) < \xi_\tau^j(0)$ implies $\xi_\tau^j(v_w) < \xi_\tau^j(\mu)$, which implies that the same cohomology class \mathfrak{T}_w^τ will be in the kernel of κ_μ for the map (6) if it is an element of the cohomology $H_T^*(Gr(2, 4))$. The classes $\alpha_1, \dots, \alpha_8$ are exactly those classes in the kernel of κ_0 which are in the image of the injection $H_T^*(Gr(2, 4)) \hookrightarrow H_T^*(SU(4)/T)$.

To compute the remaining degree 2 classes, we do as before: Consider $\tau = [4321]$. A fundamental weight minimized at $v_{[2211]}$ is $\xi = e_1 + e_2$. We note that

$$\xi(v_{[2211]}) < \xi(v_{[1212]}) = \xi(v_{[2112]}) = \xi(v_{[1212]}) = \xi(v_{[1212]}) = 0$$

whereas $\xi(\mu) = \mu_1 + \mu_2 > 0$. Note that the fixed points λ_w in \mathcal{O}_λ which map to these points in \mathcal{O}_ν under the forgetful map do not have the same property as above. When one evaluates ξ on the points λ_w , several will be above $\xi(0)$, which is why we anticipate that the corresponding \mathfrak{T}_w^τ will not have been already seen in the quotient for the generic case.

Let $w = [1324]$, which maps to $[1212]$ under the forgetful map, and calculate $\mathfrak{T}_w^\tau(x, u) = \mathfrak{T}_{\tau^{-1}w}^{id}(x, u_\tau)$. First we find $\tau^{-1}w = [4231] = s_3s_1s_2s_1s_3$:

$$\begin{aligned} \mathfrak{T}_{\tau^{-1}w}^{id}(x, u) &= \partial_{w^{-1}\tau} A \\ &= \partial_3 \partial_1 \partial_2 \partial_1 \partial_3 A \\ &= \partial_3 \partial_1 \partial_2 \partial_1 (x_1 - u_2)(x_1 - u_3)(x_1 - u_4)(x_2 - u_3)(x_2 - u_4) \\ &= \partial_3 \partial_1 \partial_2 (x_1 - u_3)(x_1 - u_4)(x_2 - u_3)(x_2 - u_4) \\ &= \partial_3 \partial_1 (x_1 - u_3)(x_1 - u_4)(x_2 + x_3 - (u_3 + u_4)) \\ &= \partial_3 (x_1 x_2 + x_2 x_3 + x_1 x_3 - (u_3 + u_4)(x_1 + x_2 + x_3) \\ &\quad + u_3^2 + u_3 u_4 + u_4^2) \\ &= (x_1 + x_2) - (u_3 + u_4). \end{aligned}$$

Then $\mathfrak{T}_w^\tau(x, u) = (x_1 + x_2) - (u_2 + u_1) = \beta_1$. Note that $\mathfrak{T}_w^\tau(x, u)$ is symmetric in x_1 and x_2 , which is necessary for it to be an element of the kernel. Similarly, one finds that there are no other degree two classes given by this choice of ξ .

For $\xi = e_2 + e_3$ and $\xi = e_1 + e_3$ and using the same techniques, one finds the classes $\beta_2 = x_1 + x_2 - (u_1 + u_4)$ and $\beta_3 = x_1 + x_2 - (u_1 + u_3)$, respectively.

One can check that the quotient ring is actually the cohomology of the two-sphere. In fact, it is known that $Gr(2, 4)//T$ is always a two-sphere for regular values of the moment map in the moment polytope.

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