Torsion in the full orbifold K-theory of abelian symplectic quotients

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Abstract   Let (M, ω, Φ) be a Hamiltonian T-space and let $H \subseteq T$ be a closed Lie subtorus. Under some technical hypotheses on the moment map Φ, we prove that there is no additive torsion in the **integral full orbifold K-theory** of the orbifold symplectic quotient $[M /\!/ H]$. Our main technical tool is an extension to the case of moment map level sets the well-known result that components of the moment map of a Hamiltonian T-space M are Morse-Bott functions on M. As first applications, we conclude that a large class of symplectic toric orbifolds, as well as certain S^1-quotients of GKM spaces, have integral full orbifold K-theory that is free of additive torsion. Finally, we introduce the notion of **semilocally Delzant** which allows us to formulate sufficient conditions under which the hypotheses of the main theorem hold. We illustrate our results using low-rank coadjoint orbits of type A and B.

Keywords K-theory · Symplectic quotient · Toric varieties · Torsion ·
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1 Introduction

The main purpose of this manuscript is to show that the **integral full orbifold K-theory** of several classes of orbifolds $\mathfrak{X}$ arising as abelian symplectic quotients are free of additive torsion. An important subclass of symplectic quotients to which our results apply are **orbifold toric varieties**, of which **weighted projective spaces** are themselves a special case.

Orbifold toric varieties are global quotients of a manifold by a torus action, and are therefore a natural starting point for a study of orbifolds. Many conjectures on orbifolds and orbifold invariants in active areas of research (algebraic geometry, equivariant topology, the theory of mirror symmetry, to name a few) have been first tested in the realm of orbifold toric varieties. More specifically, there has been historically [2–4,29,34] and also quite recently a burst of interest in weighted projective spaces (and their **integral** invariants); for more recent work, see for instance [7,9,12,17,26,37].

We note that the application of our main result to orbifold toric varieties is in the spirit of the previous work in the study of (both ordinary and orbifold) topological invariants of weighted projective spaces [3,15,29]. Moreover, recent work of Hua [27] uses algebro-geometric methods to show that Grothendieck groups of a large class of toric Deligne-Mumford stacks are free of additive torsion. This part of our results (Theorem 1.2 below) can be viewed as a full orbifold K-theory analogue of his results in the topological category, proved via symplectic geometric methods. However, the scope of our results is more general. While the above-mentioned work all deal with certain cases of orbifold toric varieties, the techniques in this manuscript, which build upon the symplectic and equivariant Morse theoretic methods developed in [15], allow us to prove that the full orbifold K-theory is free of additive torsion in more general settings. In particular, we discuss non-toric examples in the later sections.

Let T denote a compact connected abelian Lie group, i.e., a torus. Suppose (M, ω, Φ) is a Hamiltonian T-space, with moment map $\Phi : M \to \mathfrak{t}^*$. Furthermore, let $\beta : H \hookrightarrow T$ be a closed Lie subgroup (i.e. a subtorus). Let $\Phi_H : M \to \mathfrak{h}^*$ be the induced H-moment map obtained as the composition of $\Phi : M \to \mathfrak{t}^*$ with the linear projection[1] $\beta^* : \mathfrak{t}^* \to \mathfrak{h}^*$. Suppose $\eta \in \mathfrak{h}^*$ is a regular value of Φ_H, and denote by $Z := \Phi_H^{-1}(\eta) \subseteq M$ the corresponding level set. Since η is a regular value, Z is a smooth submanifold of M, and H acts locally freely on Z. Let

$$\mathfrak{X} := [M /\!/_\eta H] = [Z/H] \tag{1.1}$$

denote the quotient stack associated to the locally free H-action on Z. This is an orbifold, also referred to as a **Deligne-Mumford stack in the differentiable category**. Let $\xi \in \mathfrak{t}$ and recall that $\Phi^\xi := \langle \Phi, \xi \rangle : M \to \mathbb{R}$ denotes the corresponding component of the moment map.

The full orbifold K-theory $\mathrm{K}_{\mathrm{orb}}(\mathfrak{X})$ over $\mathbb{Q}$ was introduced by Jarvis, Kaufman and Kimura in [28]. In the case that $\mathfrak{X}$ is formed as an abelian quotient of a manifold Z by a locally free action of a torus T, the authors of this manuscript and Kimura gave an integral lift $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X})$ in terms of the inertial K-theory $N K_T(Z)$ in [15]. This satisfies $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X}) \otimes \mathbb{Q} \cong \mathrm{K}_{\mathrm{orb}}(\mathfrak{X})$ as rings [8]. Specifically, the full orbifold K-theory may be described additively as a module over $K_T(\mathrm{pt})$ by

$$\mathbb{K}_{\mathrm{orb}}(\mathfrak{X}) = \bigoplus_{t \in T} K_T(Z^t).$$

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[1] By slight abuse of notation we use β to also denote the linear map $\mathfrak{h} \to \mathfrak{t}$ obtained as the derivative of the inclusion map $\beta : H \hookrightarrow T$.

This differs from previous definitions of "orbifold K-theory," e.g. that of Adem and Ruan [1]. We refer the reader to the introduction of [15] for a more detailed discussion of other notions of orbifold K-theory in the literature. In this manuscript, for G a compact Lie group and Y a G-space, we let $K_G(Y) = K_G^0(Y)$ denote the Atiyah-Segal topological G-equivariant K-theory [35]. This is built from G-equivariant vector bundles when Y is a compact G-space, and G-equivariant maps $[Y, \text{Fred}(\mathcal{H}_G)]_G$ if Y is noncompact (here $\mathcal{H}_G$ is a Hilbert space that contains infinitely many copies of every irreducible representation of G, see e.g. [6]). We may now state our main theorem about the structure of $\mathbb{K}_{\text{orb}}(\mathfrak{X})$.

Theorem 1.1 *Let (M, ω, Φ) be a Hamiltonian T-space, $\beta : H \hookrightarrow T$ a connected subtorus with induced moment map $\Phi_H := \beta^* \circ \Phi : M \to \mathfrak{h}^*$. Suppose $\eta \in \mathfrak{h}^*$ is a regular value of Φ_H, let $Z := \Phi_H^{-1}(\eta)$ denote its level set, and $\mathfrak{X} := [Z/H]$ the associated quotient orbifold stack. Suppose there exists $\xi \in \mathfrak{t}$ such that the following conditions hold:*

(1) $H \subseteq \overline{\exp(t\xi)}$, *the closure of the one-parameter subgroup generated by ξ in T;*
(2) $f := \Phi^\xi|_Z$ *is proper and bounded below;*
(3) *for each $t \in H$, $\pi_0(\text{Crit}(f|_{Z^t}))$ is finite;*
(4) *for each $t \in H$ and each connected component C of $\text{Crit}(f|_{Z^t})$,*

    (a) $K_H^0(C)$ *contains no additive torsion, and*
    (b) $K_H^1(C) = 0$.

Then $\mathbb{K}_{\text{orb}}(\mathfrak{X})$ contains no additive torsion.

A direct consequence is that when the components of the critical set are isolated H-orbits, $\mathbb{K}_{\text{orb}}(\mathfrak{X})$ contains no additive torsion (see Corollary 3.2). We use this corollary to prove the following.

Theorem 1.2 *Let $\mathfrak{X}$ be a symplectic toric orbifold obtained as a symplectic quotient of a linear H-action on a complex affine space, where H is a connected compact torus. Then $\mathbb{K}_{\text{orb}}(\mathfrak{X})$ is free of additive torsion.*

As mentioned above, this corollary is similar in spirit to Kawasaki's result that the integral cohomology of (the underlying topological spaces of) weighted projective spaces are free of additive torsion. Kawasaki showed in [29] that the integral cohomology groups of (the coarse moduli space of) a weighted projective space agree with those of a smooth projective space, but the ring structure differs, with structure constants that depend on the weights. Hence we expect that the richness of the data in $\mathbb{K}_{\text{orb}}(\mathfrak{X})$ for toric orbifolds is also contained not in additive torsion but rather in the multiplicative structure constants of the ring. We leave this for future work.

The main theorem may be applied in situations other than that of the Delzant construction of orbifold toric varieties; the content of the last two sections of this manuscript is an exploration of other situations in equivariant symplectic geometry in which the hypotheses also hold. First, we observe in Sect. 5 that the hypotheses above on the relevant connected components C hold for S^1-symplectic quotients of Hamiltonian T-spaces which are GKM, under a technical condition on the choice of subgroup $S^1 \subseteq T$. Spaces with T-action which satisfy the so-called "GKM conditions," introduced in the influential work of Goresky-Kottwitz-Macpherson [16], are extensively studied in equivariant algebraic geometry, symplectic geometry, and geometric representation theory, and encompass a wide array of examples. We use a corollary of the main theorem to prove Theorem 5.1, which states that the full orbifold K-theory of the quotient of a GKM space by certain circle subgroups is free of additive torsion. Secondly, we

explore in Sect. 6 how the essential properties of the Delzant construction (which allow us to prove Theorem 1.2) may in fact be placed in a more general framework of phenomena in torus-equivariant symplectic geometry which may be informally described as 'taking place within a T-equivariant Darboux neighborhood of an isolated T-fixed point.' The precise statements are given in detail in Sect. 6, where we introduce the notion of a closed H-invariant subset of a Hamiltonian T-space being **semilocally Delzant** (with respect to H), and make some initial remarks on situations in which this notion applies. One class of spaces to which our definitions apply are the generalized flag varieties G/B and G/P, which may be covered by Darboux neighborhoods given by the Weyl translates of the open Bruhat cell. Furthermore, the natural T-action on generalized flag varieties (that of the maximal torus T in G) is also well-known to be GKM. In both Sects. 5 and 6, we illustrate our results using examples of this type.

2 A local normal form and Morse-Bott theory on level sets of moment maps

We begin with our main technical lemma (Lemma 2.2) regarding the Morse-Bott theory of moment maps in equivariant symplectic geometry. The techniques used to prove this result are fairly standard in the field, but we have not seen this particular formulation in the literature. It is well-known that components of moment maps $\Phi^\xi = \langle \Phi, \xi \rangle : M \to \mathbb{R}$ are Morse-Bott functions on a Hamiltonian T-space M, for any $\xi \in \mathfrak{t}$. In addition, these components induce Morse-Bott functions on smooth symplectic quotients $M /\!/_\eta H$, where H is a closed Lie subgroup of T, and η is a regular value of the H-moment map Φ_H [33].

What seems heretofore unnoticed[2] is that a component Φ^ξ of the T-moment map, restricted to the level set $\Phi_H^{-1}(\eta)$ itself, is also a Morse-Bott function when H is contained in the closure of the subgroup generated by ξ. This may be deduced from the following local normal form result of Hilgert, Neeb, and Plank [25, Lemmata 2.1 and 2.2], which builds on work of Guillemin and Sternberg [19, Chapter II]. Note that generic ξ satisfy this condition.

Proposition 2.1 (Hilgert, Neeb, Plank) *Let* (M, ω, Φ) *be a Hamiltonian T-space with moment map* $\Phi : M \to \mathfrak{t}^*$. *Let* $p \in M$. *Then there exists a T-invariant neighborhood* $\mathcal{U} \subseteq M$ *of the orbit* $T \cdot p \subseteq M$, *a subtorus* $T_1 \subseteq T$ *and a symplectic vector space V such that:*

1. *There is a decomposition* $T = T_0 \times T_1$, *where* $T_0 = \mathrm{Stab}(p)_0$ *is the connected component of the identity in the stabilizer group of p in T.*
2. *There is a T-equivariant symplectic open covering from an open subset* $\mathcal{U}'$ *of* $T_1 \times \mathfrak{t}_1^* \times V$ *onto* $\mathcal{U}$, *where the T-action on* $T_1 \times \mathfrak{t}_1^* \times V$ *is given by*

$$(T_0 \times T_1) \times (T_1 \times \mathfrak{t}_1^* \times V) \to (T_1 \times \mathfrak{t}_1^* \times V)$$
$$((t_0, t_1), (g, \gamma, v)) \mapsto (t_1 \cdot g, \gamma, \rho(t_0)v),$$

(2.1)

where $\rho : T_0 \to Sp(V)$ *is a linear symplectic representation.*

3. *There exists a complex structure I on V such that* $\langle v, w \rangle := \omega_V(Iv, w)$ *defines a positive definite scalar product on V. Let* $V = \bigoplus_\alpha V_\alpha$ *be the decomposition of V into isotypic components corresponding to weights* $\alpha \in \mathfrak{t}_0^*$. *With respect to these local coordinates,*

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[2] However, a result of this nature appears to be implicit in the work of Lerman and Tolman on the classification of orbifold toric varieties [32], and even earlier in work of Marsden and Weinstein [33] and Atiyah [5].

the moment map Φ' on $\mathcal{U}' \subseteq T_1 \times \mathfrak{t}_1^* \times V$ is given by

$$\Phi' : \mathcal{U}' \subseteq T_1 \times \mathfrak{t}_1^* \times V \to \mathfrak{t}^* \cong \mathfrak{t}_0^* \oplus \mathfrak{t}_1^*$$

$$(g, \eta, v) \mapsto \Phi'(1, 0, 0) + \left(\frac{1}{2} \sum ||v_\alpha||^2 \alpha, \eta \right). \tag{2.2}$$

For any $\xi \in \mathfrak{t}$, let $T^\xi := \overline{\exp(t\xi)}$ denote the closure of the one-parameter subgroup generated by $\xi \in \mathfrak{t}$. Using the notation set in the Introduction, we now have the following.

Lemma 2.2 *Let (M, ω, Φ) be a Hamiltonian T-space, and $H \subseteq T$ a subtorus. Let $Z := \Phi_H^{-1}(\eta)$ be a level set of the moment map for the H action at a regular value. The function*

$$f := \Phi^\xi|_Z : Z \to \mathbb{R}$$

is a Morse-Bott function on Z for every $\xi \in \mathfrak{t}$ such that $H \subseteq T^\xi$.

Proof We show that for any point $p \in Z$ such that $df_p = 0$,

1. the connected component of $\mathrm{Crit}(f)$ containing p is a submanifold, where $\mathrm{Crit}(f)$ is the critical set of f, and
2. the Hessian of f at p is non-degenerate in the directions normal to the connected component of $\mathrm{Crit}(f)$ containing p.

Since the conditions to be checked are purely local, we may argue separately for each point p in the critical set $\mathrm{Crit}(f)$.

For the purposes of this argument, we may assume without loss of generality that the T-equivariant symplectic open cover $\mathcal{U}' \to \mathcal{U}$ of Proposition 2.1 is in fact a T-equivariant symplectomorphism. The only part of this claim requiring justification is the relationship, in general, between the moment maps Φ_1 and Φ_2 associated to Hamiltonian T-spaces (M_1, ω_1, Φ_1) and (M_2, ω_2, Φ_2) where there exists a T-equivariant symplectic open cover $\pi : M_1 \to M_2$. Since by assumption $\pi_1^* \omega_2 = \omega_1$ and $\pi_*(\xi_{M_1}^\sharp) = \xi_{M_2}^\sharp$ for all $\xi \in \mathfrak{t}$ where $\xi_{M_i}^\sharp$ denotes the infinitesmal vector fields generated by ξ on the M_i, it follows immediately from Hamilton's equations that $\pi^* \Phi_2$ may be chosen as a moment map Φ_1 for the T-action on M_1. In particular, since π is an open covering, the local argument for Φ_1 in a small enough neighborhood of a point p in M_1 translates directly to an analogous argument in M_2 for Φ_2. Therefore we henceforth assume that (2.1) and (2.2) locally represent a neighborhood of p, and Φ near $p \in Z$, respectively.

We continue with a characterization of the critical points $\mathrm{Crit}(f) \subseteq Z$. Recall $T^\xi := \overline{\exp(t\xi)}$. Let $\mathrm{Stab}_{T^\xi}(p)$ denote the stabilizer group in T^ξ of p and $\mathrm{codim}(H, T^\xi)$ the codimension of the subgroup H in T^ξ. Suppose $p \in Z$. We claim that $p \in \mathrm{Crit}(f)$ if and only if $\dim(\mathrm{Stab}_{T^\xi}(p)) = \mathrm{codim}(H, T^\xi))$. Note that $p \in Z$ immediately implies $\dim \mathrm{Stab}_{T^\xi}(p) \leq \mathrm{codim}(H, T^\xi))$, since H acts locally freely on Z. By definition, a point $p \in Z$ is critical for f if and only if

$$df_p(v) = \langle d\Phi_p(v), \xi \rangle = \omega_p(\xi_p^\sharp, v) = 0, \quad \forall v \in \mathsf{T}_p Z,$$

where $\mathsf{T}_p Z$ denotes the tangent space at p to Z. Note also that the tangent space

$$\mathsf{T}_p Z = \mathsf{T}_p \Phi_H^{-1}(\eta) = (\mathsf{T}_p(H \cdot p))^{\omega_p} \subseteq \mathsf{T}_p M.$$

Thus $p \in Z$ is critical for f if and only if

$$\xi_p^\sharp \in ((\mathsf{T}_p(H \cdot p))^{\omega_p})^{\omega_p} = \mathsf{T}_p(H \cdot p).$$

Since ξ generates T^ξ, it follows that $p \in \mathrm{Crit}(f)$ if and only if

$$\mathsf{T}_p(T^\xi \cdot p) \subseteq \mathsf{T}_p(H \cdot p). \tag{2.3}$$

Hence $\dim \mathrm{Stab}_{T^\xi}(p) \geq \mathrm{codim}(H, T^\xi)$. Thus $p \in Z$ is critical for f if and only if $\dim \mathrm{Stab}_{T^\xi}(p) = \mathrm{codim}(H, T^\xi)$.

The above argument shows that for any $\xi \in \mathfrak{t}$ with $H \subseteq T^\xi$, the critical set $\mathrm{Crit}(f)$ is precisely the union of sets of the form $Z^{(T')}$ for subtori T' of T^ξ such that $\dim(T') = \mathrm{codim}(H, T^\xi)$, where

$$Z^{(T')} := \{p \in Z : \mathrm{Stab}_{T^\xi}(p) = T'\}$$

consists of the points whose stabilizer group in T^ξ is precisely T'. Since H acts locally freely on Z, a subtorus T' of T^ξ as above has maximal dimension among subtori of T^ξ with nonempty $Z^{(T')}$. Now let $p \in \mathrm{Crit}(f)$. Consider local coordinates near p as in (2.1), with Φ near p described by (2.2). Write $\xi = \xi_0 + \xi_1$ for $\xi_0 \in \mathfrak{t}_0, \xi_1 \in \mathfrak{t}_1$. We first determine the intersection of $\mathrm{Crit}(f)$ with this coordinate chart, in terms of these local coordinates. From the description of the $T = T_0 \times T_1$-action in (2.1), and from the fact observed above that p is in $\mathrm{Crit}(f)$ precisely when its stabilizer subgroup is of maximal possible dimension, it follows that $\mathrm{Crit}(f)$ is the set of points of the form $\{(g, \gamma, v) : v \in V_0\}$ where V_0 is the subspace of V on which T_0 acts trivially. In particular, $\mathrm{Crit}(f)$ is a submanifold of Z near p.

Finally, we show that the Hessian of f near p is nondegenerate on those tangent directions in $\mathsf{T}_p Z$ corresponding to tangent vectors of the form $\{(0, 0, \sum_{\alpha \neq 0} v_\alpha) : v_\alpha \in V_\alpha, \alpha \neq 0\}$ in the chosen local coordinates. Recall that for tangent vectors $v, w \in \mathsf{T}_p Z$, the Hessian $\mathrm{Hess}(f)_p(v, w)$ is computed by $\mathcal{L}_{\tilde{v}}\mathcal{L}_{\tilde{w}}(f)$ where $\tilde{v}, \tilde{w}$ are arbitrary extensions of v, w to vector fields in a neighborhood of p in Z (and $\mathcal{L}_X$ denotes a Lie derivative along a vector field X). In the local coordinates of Proposition 2.1, any two vectors of the form $v = \left(0, 0, \sum_{\alpha \neq 0} v_\alpha\right), w = \left(0, 0, \sum_{\alpha \neq 0} w_\alpha\right)$ may be extended to a neighborhood as the constant vector field $\tilde{v} \equiv \left(0, 0, \sum_{\alpha \neq 0} v_\alpha\right), \tilde{w} \equiv \left(0, 0, \sum_{\alpha \neq 0} w_\alpha\right)$. We then observe that the description of Φ in (2.2) implies that for such a $\tilde{w}$,

$$\mathcal{L}_{\tilde{w}}(f) = df(\tilde{w}) = d(\Phi^{\xi_0}|_Z)(\tilde{w}),$$

since $\tilde{w}$ contains no component in $\mathfrak{t}_1^*$. It then suffices to show that the Hessian of the $\mathfrak{t}_0^*$-component of Φ is nondegenerate in the directions $\oplus_{\alpha \neq 0} V_\alpha$. From the local normal form of Φ in (2.2), this is just a standard quadratic moment map for a linear symplectic action of a torus on a symplectic vector space, so this non-degeneracy is classical (see e.g. [5]). $\quad\square$

3 The proof and a corollary of the main theorem

We now prove the main theorem. The argument uses equivariant Morse theory of the moment map, most of which is standard (see, for example, [24,30,31,36]). The novel feature here involves the use of a component of the moment map on a level set of a moment map for a partial torus action. We use the same notation as in the introduction.

Proof of Theorem 1.1 We first note that since the statement of the theorem involves only the additive structure of $\mathbb{K}_{\mathrm{orb}}$, we need only recall the definition (and computation) of $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X})$ as an additive group. In [15] (cf. also [8]), the integral full orbifold K-theory of orbifolds

$\mathfrak{X}$ arising as abelian symplectic quotients (by a torus H) is described via an isomorphism [15, Remark 2.5]

$$\mathbb{K}_{\mathrm{orb}}(\mathfrak{X}) \cong NK_H(Z) := \bigoplus_{t \in H} K_H\left(Z^t\right)$$

where the middle term is the H-equivariant integral **inertial K-theory** of the manifold $Z := (\Phi_H)^{-1}(\eta)$, defined additively as the direct sum above. We now show that the right-hand side is torsion free.

Note that $Z^t = (\Phi_H|_{M^t})^{-1}(\eta)$, so it is itself a level set for the H-moment map on M^t for each $t \in T$. Suppose $\xi \in \mathfrak{t}$ satisfies the hypotheses of the theorem, and let $f = \Phi^{\xi}|_Z$. Since f is proper and bounded below, then clearly $f|_{Z^t}$ is also proper and bounded below. It is now immediate that ξ satisfies conditions (1)–(4) for the Hamiltonian T-space M^t. Thus without loss of generality, we need only check that $K_H(Z)$ is torsion-free; all other cases follow similarly.

By Lemma 2.2, f is a Morse-Bott function. Denote the connected components of $\mathrm{Crit}(f)$ by $\{C_j\}_{j=1}^{\ell}$, where ℓ is finite by condition (3) and assume without loss of generality that $f(C_i) < f(C_j)$ if $i < j$. Because f is bounded below and proper, all components are closed and compact, and there exists a minimal component, which we denote C_0.

Assume Z is nonempty. We build the equivariant K-theory of Z inductively by studying the critical sets, beginning with the base case. By assumption, $K_H^0(C_0)$ has no additive torsion and $K_H^1(C_0) = 0$. For small enough $\varepsilon > 0$, consider the submanifolds

$$Z_j^+ = f^{-1}((-\infty, f(C_j) + \varepsilon)), \quad Z_j^- = f^{-1}((-\infty, f(C_j) - \varepsilon)),$$

where ε is chosen so that C_j is the only critical component contained in $Z_j^+ \setminus Z_j^-$. Using the 2-periodicity of (equivariant) K-theory, there is a periodic long exact sequence

$$K_H^0(Z_j^+) \longrightarrow K_H^0(Z_j^-)$$

$$K_H^0(Z_j^+, Z_j^-) \qquad\qquad K_H^1(Z_j^+, Z_j^-) \qquad (3.1)$$

$$K_H^1(Z_j^-) \longleftarrow K_H^1(Z_j^+)$$

in equivariant K-theory for the pair (Z_j^+, Z_j^-). Choose an H-invariant metric on Z, and identify $K_H^*(Z_j^+, Z_j^-)$ with $K_H^*(D(\nu_j^-), S(\nu_j^-))$, where $D(\nu_j^-), S(\nu_j^-)$ are the disc and sphere bundles, respectively, of the negative normal bundle to C_j with respect to f. The equivariant Thom isomorphism also says that $K_H^*(D(\nu_j^-), S(\nu_j^-)) \cong K_H^*(C_j)$. There is no degree shift since the (real) dimension of the negative normal bundle is even (as can be seen from Proposition 2.1) and K-theory is 2-periodic. By assumption, $K_H^1(C_j) = 0$, and by the inductive assumption we have $K_H^1(Z_j^-) = 0$. Hence we may immediately conclude from (3.1) that $K_H^1(Z_j^+) = 0$ and that there is a short exact sequence

$$0 \to K_H^0(Z_j^+, Z_j^-) \to K_H^0(Z_j^+) \to K_H^0(Z_j^-) \to 0. \qquad (3.2)$$

By induction, $K_H^0(Z_j^-)$ has no additive torsion, and by assumption,

$$K_H^0(Z_j^+, Z_j^-) \cong K_H^0(D(\nu_j^-), S(\nu_j^-)) \cong K_H^0(C_j)$$

does not either. We conclude that $K_H^0(Z_j^+)$ is also free of additive torsion. Hence by induction we conclude that $K_H^0(Z_\ell^+)$ is free of additive torsion. Since C_ℓ is the maximal critical component, there are no higher critical sets, so the negative gradient flow with respect to f yields an H-equivariant deformation retraction from Z to Z_ℓ^+. Hence $K_H(Z) \cong K_H(Z_\ell^+)$, and in particular we may conclude that $K_H^0(Z)$ is free of additive torsion, as desired. □

Remark 3.1 In the course of the proof, we have also shown that $K_H^1(Z^t) = 0$ for all $t \in H$. In the inductive arguments given in Sects. 4 and 5, we will need this additional fact to obtain Theorems 4.1 and 5.1.

We now turn to the first application of Theorem 1.1, the case when the critical set consists of isolated H-orbits.

Corollary 3.2 *Let $\mathfrak{X} = [Z/H]$ be an orbifold constructed as in (1.1). As above, suppose that there exists $\xi \in \mathfrak{t}$ such that*

- $H \subseteq T^\xi$,
- $f := \Phi^\xi|_Z$ *is proper and bounded below, and*
- *for every $t \in H$, $Crit(f|_{Z^t})$ consists of finitely many isolated H-orbits.*

Then $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X})$ contains no additive torsion. Furthermore, $K_H^1(Z^t) = 0$ for all $t \in H$.

Proof It suffices to check that the hypotheses of Theorem 1.1 are satisfied, and it is evident that the only assumption needing comment is (4). Since each connected component C is an isolated H-orbit, and by assumption H acts locally freely on Z, we have

$$K_H^0(C) \cong K_H^0(H \cdot p) \cong K_H^0(H/\Gamma),$$

where $p \in C$ and Γ is the finite stabilizer subgroup $\mathrm{Stab}_T(p)$ in H. The H-equivariant K-theory of a homogeneous space is the representation ring of the stabilizer of the identity coset,

$$K_H^0(H/\Gamma) \cong K_\Gamma^0(\mathrm{pt}) \cong R(\Gamma),$$

which has no additive torsion. Moreover, $K_H^1(H/\Gamma) \cong K_\Gamma^1(\mathrm{pt}) = 0$. Hence, assumptions (4a) and (4b) hold, and we may apply the Main Theorem. The result follows. □

This corollary provides the starting point for inductive arguments which show that the integral full orbifold K-theory of an abelian symplectic quotient is torsion free.

Remark 3.3 It follows immediately from this proof that the integral full orbifold K-theory $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X})$ of an orbifold $\mathfrak{X} = [Z/H]$ satisfying the hypotheses of the Main Theorem is additively the direct sum of representation rings $R(\Gamma)$ for those subgroups Γ of H appearing as stabilizer groups in the level set of the moment map $Z = \Phi_H^{-1}(\eta)$. It would be interesting to compare this description via representation rings to the computation given in [15] in terms of the Kirwan surjectivity theorem in full orbifold K-theory.

4 Symplectic toric orbifolds

We now provide a first application of the Main Theorem and its corollary, namely: for a large class of toric orbifolds, the integral full orbifold K-theory contains no additive torsion. In the case of weighted projective spaces similar results were obtained by Kawasaki in ordinary

integral cohomology in the 1970s [29], then in ordinary K-theory (using results of [29]) by Al Amrani in [3]. More recently, Zheng Hua [27] has independently shown using algebro-geometric methods that, when the generic point is stacky, the Grothendieck group $K_0(\mathfrak{X}_\Sigma)$ of a smooth complete toric Deligne-Mumford stack is a free $\mathbb{Z}$-module. Here, Σ is a stacky fan as defined in [10] and K_0 is the algebraic K-theory defined via coherent sheaves. Since it is straightforward to see from the definition (given below) of symplectic toric orbifolds $\mathfrak{X}$ that the twisted sectors arising in the computation of the full orbifold K-theory $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X})$ are themselves stacks which are symplectic toric orbifolds, the substantive statement (which is the topological K-theory analogue of Hua's result) is that each twisted sector individually has K-theory free of additive torsion. Hence Theorem 4.1 should be viewed as a straightforward integral full orbifold K-theory analogue, in the topological category and for symplectic toric orbifolds $\mathfrak{X}$, of Hua's result [27]. However, our methods of proof, which use the equivariant Morse theory of symplectic geometry developed in Sects.2 and 3, are significantly different from those of [27].

We first establish notation for both the Delzant construction of toric varieties and the statement of the theorem. In the smooth case, this construction may be found in [13] (for an accessible account, see [11]). This construction is generalized to the orbifold case in [32]. Let $T^n = (S^1)^n$ be the standard compact n-torus, acting in the standard linear fashion on $\mathbb{C}^n$ (via the embedding of T^n into $U(n, \mathbb{C})$ as diagonal matrices with unit complex entries). This is a Hamiltonian T^n-action on $\mathbb{C}^n$ with respect to the standard Kähler structure on $\mathbb{C}^n$.

Let $\Phi : \mathbb{C}^n \to (\mathfrak{t}^n)^*$ denote a moment map for this action. For a connected closed subtorus $\beta : H \hookrightarrow T^n$, let $\Phi_H := \beta \circ \Phi : \mathbb{C}^n \to \mathfrak{h}^*$ denote the induced moment map. For a regular value $\eta \in \mathfrak{h}^*$ of Φ_H, let $Z := \Phi_H^{-1}(\eta)$ be its level set. By regularity of η, H acts locally freely on Z. The **symplectic toric orbifold** specified by $\beta : H \hookrightarrow T^n$ and η is then defined by

$$\mathfrak{X} := \mathbb{C}^n /\!\!/_\eta H = [Z/H].$$

The procedure just recounted is often called the **Delzant construction** of the toric orbifold $\mathfrak{X}$, although historically it was the underlying topological space of $\mathfrak{X}$ that was studied, not the associated stack.[3] Symplectic toric orbifolds were classified in [32]; we consider only those obtained by a quotient by a connected subtorus H. We will call an element $\xi \in \mathfrak{t}$ of the Lie algebra **generic** if its associated 1-parameter subgroup $\exp(t\xi)$ in T is dense: in the notation of Sect. 2, $T^\xi = T$. Note that if there exists any $\xi \in \mathfrak{t}$ such that $\Phi^\xi|_Z$ is proper and bounded below, then there also exists a generic $\xi \in \mathfrak{t}$ satisfying the same conditions.

Theorem 4.1 *Let $\mathfrak{X} = \mathbb{C}^n /\!\!/_\eta H$ be a symplectic toric orbifold, where $\beta : H \hookrightarrow T$ a connected closed subtorus of T and $\eta \in \mathfrak{h}^*$ a regular value. Let $Z = \Phi_H^{-1}(\eta)$ denote the η-level set of Φ_H. Then $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X})$ has no additive torsion. Furthermore, $K_H^1(Z^t) = 0$ for all $t \in H$.*

Proof Since the original T-action on $\mathbb{C}^n$ is a standard linear action by diagonal matrices, for any $t \in H$, the fixed point set $(\mathbb{C}^n)^t$ is a coordinate subspace, i.e. $(\mathbb{C}^n)^t \cong \mathbb{C}^m \subset \mathbb{C}^n$, determined by the values of the T-weights on each coordinate line $\{(0, 0, \ldots, z_j, 0, \ldots, 0)\} \subseteq \mathbb{C}^n$. It is immediate that $(\mathbb{C}^n)^t$ is a linear symplectic subspace of $\mathbb{C}^n$ and that the restriction $\Phi_H|_{(\mathbb{C}^n)^t} : (\mathbb{C}^n)^t \to \mathfrak{h}^*$ is a moment map for this action. Thus Z^t is equal to $\left(\Phi_H|_{(\mathbb{C}^n)^t}\right)^{-1}(\eta)$, the level set of a moment map for an H-action on a possibly-smaller-dimensional vector space.

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[3] Indeed, the underlying topological space Z/H corresponding to the stack $\mathfrak{X}$ is often also called the **symplectic quotient of $\mathbb{C}^n$ by H at the value** η. In the current literature, there is an unfortunate ambiguity: the "symplectic quotient" may refer to the stack or the underlying topological space.

Choose a generic $\xi \in \mathfrak{t}$ such that $\Phi^\xi|_Z$ is proper and bounded below. Such a ξ exists because there are such components for $\Phi : \mathbb{C}^n \to \mathfrak{t}^*$, and Z is a T-invariant closed subset of $\mathbb{C}^n$. Let $f = \Phi^\xi|_Z$. In order to apply Corollary 3.2, we must check that for all $t \in H$, the critical set $\text{Crit}(f|_{Z^t})$ consists of finitely many isolated H-orbits. We first observe that since $(\mathbb{C}^n)^t \cong \mathbb{C}^m$ is itself a symplectic linear space equipped with a linear T-action, it suffices to prove this statement for the special case $t = \text{id}$; the other cases follow similarly.

Let C be a connected component in $\text{Crit}(f)$ and $p \in C$. Since f is T-invariant, $H \cdot p \subset C$. Since C is compact and connected, it suffices to show that C consists of one orbit locally. Recall from the proof of Lemma 2.2 that $p \in \text{Crit}(f)$ exactly if

$$\dim(\text{Stab}_T(p)) = \text{codim}(H) = n - k.$$

Thus $\dim(\text{Stab}_T(p)) = n - k$ exactly if $p = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ has precisely $n - k$ coordinates equal to 0, i.e. p lies in a coordinate subspace of $\mathbb{C}^n$ isomorphic to $\mathbb{C}^k$. Note that H acts on $\mathbb{C}^n$ preserving this $\mathbb{C}^k$, and the regularity assumption on η implies that the restriction of Φ_H to $\mathbb{C}^k$ is $\mathbb{Q}$-linearly isomorphic to the standard moment map for the standard H-action on $\mathbb{C}^k$ (up to a translation by a constant in $\mathfrak{h}^*$). In particular, this implies that the condition $p \in Z := \Phi_H^{-1}(\eta)$ for a regular value η uniquely determines the non-zero norms of the coordinates $\|z_{i_1}\|^2, \|z_{i_2}\|^2, \ldots, \|z_{i_k}\|^2$. Therefore, the only nearby points $p' \in Z$ with $\dim(\text{Stab}_T(p')) = n - k$ are those in the H orbit of p. We conclude that each connected component C of $\text{Crit}(f)$ is a single H-orbit. Moreover, there are only finitely many critical components because there are only finitely many k-dimensional coordinate subspaces of $\mathbb{C}^n$.

The same argument for each $(\mathbb{C}^n)^t$ and an application of Corollary 3.2 completes the proof. $\qquad\square$

5 GKM spaces

Let (M, ω, Φ) be a compact Hamiltonian T-space. Suppose in addition that the T-fixed points are isolated, and that the set of points with codimension-1 stabilizer

$$M_1 := \{x \in M \mid \text{codim}(\text{Stab}(x)) = 1\}$$

has real dimension $\dim(M_1) \leq 2$. When these conditions are satisfied, we say that M is a **GKM space** and that the T-action on M is **GKM**.[4] It is also well-known in the theory of GKM spaces (in the context of the study of Hamiltonian T-actions) that these conditions imply that the **equivariant 1-skeleton of** M, i.e. the closure $\overline{M_1} = M_1 \cup M^T$, is a union of symplectic 2-spheres S^2. Moreover, each such 2-sphere is itself a Hamiltonian T-space; the T-action on S^2 is given by a nontrivial character $T \to S^1$ (equivalently, a nonzero weight $\alpha \in \mathfrak{t}_\mathbb{Z}^*$) where the S^1 acts on S^2 by rotation. Here the weight α is obtained from the linear T-isotropy data at either of the two T-fixed points in S^2. (For details see e.g. the expository article [38].)

Hamiltonian T-spaces (M, ω, Φ) (or algebraic varieties equipped with algebraic torus actions) for which the T-action is GKM have been extensively studied in modern equivariant symplectic and algebraic geometry, primarily due to the link provided by GKM theory between T-equivariant topology and the combinatorics of what is called the **moment graph** (or **GKM graph**) of M. Many natural examples arise in the realm of geometric representation theory and Schubert calculus, including generalized flag varieties G/B and G/P of

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[4] There are many variants on the definition of GKM actions (see e.g. [20–23]). In particular, in less restrictive versions, the T-space M need not be compact nor symplectic, nor even finite-dimensional.

Kac-Moody groups G (where B is a Borel subgroup and, more generally, P a parabolic subgroup). Hence the orbifold invariants of the orbifold symplectic quotients of GKM spaces is a natural area of study.

If the T-action is GKM, then for a large class of circle subgroups of T, the associated orbifold symplectic quotients $M /\!/_\eta S^1$ have no additive torsion in full integral orbifold K-theory, as we now see.

Theorem 5.1 *Suppose that (M, ω, T, Φ) is a compact Hamiltonian T-space, and suppose further that the T-action is GKM. Suppose that $\beta : S^1 \hookrightarrow T$ is a circle subgroup in T such that $M^{S^1} = M^T$, and let $\Phi_{S^1} := \beta^* \circ \Phi : M \to \mathrm{Lie}(S^1)^*$ denote the induced moment map. Let $\eta \in \mathrm{Lie}(S^1)$ be a regular value of Φ_{S^1}, and $\mathfrak{X} = M /\!/_\eta S^1$ the orbifold symplectic quotient. Then $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X})$ is free of additive torsion.*

Proof We show that the hypotheses of Corollary 3.2 hold. Let $Z := \Phi_{S^1}^{-1}(\eta)$, choose $\xi \in \mathfrak{t}$ such that its 1-parameter subgroup in T is dense in T, and let $f := \Phi^\xi|_Z$. Properness of f is immediate since M is compact. Hence it suffices to show that the critical sets $\mathrm{Crit}(f)$ and $\mathrm{Crit}(f|_{Z'})$ are isolated S^1-orbits. Observe that when M is a GKM space, M^t is also a GKM space for any $t \in S^1$. Hence it suffices to argue only for the case of $\mathrm{Crit}(f)$; the others follow similarly.

By the argument given in the proof of Lemma 2.2, $\mathrm{Crit}(f)$ consists precisely of those points $p \in Z$ satisfying $\mathrm{codim}(\mathrm{Stab}_T(p)) = 1$. In other words, $\mathrm{Crit}(f) = Z \cap M_1$. The closure $\overline{M_1}$ consists of a union of 2-spheres, and the T-action on each S^2 is specified by a non-zero weight $\alpha \in \mathfrak{t}_{\mathbb{Z}}^*$ obtained from the T-isotropy decomposition at one of the two fixed points of the S^2. By assumption on the circle subgroup S^1, the kernel of the character $\phi_\alpha : T \to S^1$ specified by α does not contain S^1. Therefore, S^1 acts nontrivially on each $S^2 \subseteq \overline{M_1}$, implying $\Phi_{S^1}|_{S^2}$ is nontrivial, and $\Phi_{S^1}^{-1}(\eta) \cap S^2$ consists of a single S^1-orbit. (Note that Z does not contain any 0-dimensional orbits of S^1 since, by assumption on regularity of η, S^1 acts locally freely on Z.)

Thus the hypotheses of Corollary 3.2 are satisfied, so $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X})$ is additively torsion-free. $\qquad\square$

Remark 5.2 We restrict to the case of compact symplectic manifolds in this section for sake of brevity. However, the arguments given above could be altered to prove analogous results in less-restrictive contexts of GKM theory (see e.g. [22,23]).

Remark 5.3 It may be an interesting exercise to generalize Theorem 5.1 to symplectic quotients of GKM spaces by higher dimensional tori. One approach would be to consider quotients of a **k-independent GKM space** (cf. [20]) by a $(k-1)$-dimensional torus.

We now illustrate use of Theorem 5.1 for some coadjoint orbits of low-rank Lie type. We will analyze examples derived from the natural G-action on coadjoint orbits of G, but we must be careful to avoid the possibility of non-effective actions (so the symplectic quotient is an **effective** orbifold). Therefore, in Examples 5.4, 5.5, and 6.4, we use the quotient group $PG := G/Z(G)$ where $Z(G)$ denotes the (finite) center of G; by slight abuse of notation, we also notate by T the image of the usual maximal torus under the quotient $G \to PG$.

Example 5.4 Let $M = \mathcal{O}_\lambda \cong \mathcal{F}lags(\mathbb{C}^3)$ be a full coadjoint orbit of the Lie group $PSU(3, \mathbb{C})$ with maximal torus T given by the standard diagonal subgroup. Here $\lambda \in \mathfrak{t}^* \subseteq \mathfrak{su}(3)^*$ and $\mathcal{O}_\lambda$ is the λ-orbit of $PSU(3)$ with respect to the usual coadjoint action. Equip $M = \mathcal{O}_\lambda$ with the Kostant-Kirillov-Souriau form ω_λ and let $\Phi : \mathcal{O}_\lambda \to \mathfrak{t}^*$ be the

Fig. 1 In grey, we indicate the image of the equivariant 1-skeleton of M. The T-fixed points correspond to the six (*corner*) vertices of the graph. The *black line* intersecting the polytope represents the moment image of the level set Z of an S^1-moment map Φ_{S^1}. There are 5 critical components C_i in Crit(f), corresponding to the 5 *thick black dots* (the images of the C_i under Φ)
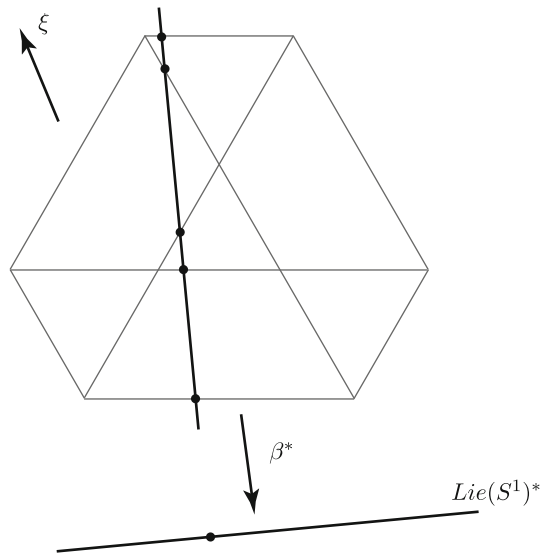
T-moment map obtained by composing the projection $\pi : \mathfrak{su}(3, \mathbb{C})^* \to \mathfrak{t}^*$ with the inclusion $\mathcal{O}_\lambda \hookrightarrow \mathfrak{su}(3, \mathbb{C})^*$. It is well-known that the T-action on M is GKM, and that the equivariant 1-skeleton of $\mathcal{O}_\lambda$ maps under Φ to the GKM graph pictured in grey in Fig. 1.

For a choice of $\beta : S^1 \hookrightarrow T$ such that $\mathcal{O}_\lambda^{S^1} = \mathcal{O}_\lambda^T$, the level set Z of the S^1-moment map $\Phi_{S^1} = \beta^* \circ \Phi$ is schematically indicated in Fig. 1 by the thick black line; the (images under Φ of the) components of Crit(f) for a generic choice of $f = \Phi^\xi|_Z$ are indicated by the thick black dots.

The standard maximal-torus T-action on a coadjoint orbit of a compact connected Lie group G is GKM; hence we may apply Theorem 5.1. From Fig. 1 we see that, additively, $NK_{S^1}(Z) = \mathbb{K}_{\mathrm{orb}}([Z/S^1])$ is a direct sum of representation rings $R(\Gamma_{i,t})$, one for each critical component $C_{i,t}$ in Crit($f|_{Z^t}$), as t ranges in S^1. In fact, only finitely many $t \in S^1$ will contribute nontrivial summands. Here the subgroup $\Gamma_{i,t}$ of S^1 is the finite stabilizer group of a point p in $C_{i,t}$, which in turn may be computed in a straightforward manner by analyzing the intersection of the chosen S^1 with each of the stabilizer subgroups appearing in the T-orbit stratification of $\mathcal{O}_\lambda$ (cf. [18, Appendix B]).

Example 5.5 Now we consider the Lie type B_2. Here we find it convenient to work with the complex form $PSO(5, \mathbb{C})$. We recall that the maximal torus T of type B_2 is 2-dimensional and the roots are given as in Fig. 2. We consider a coadjoint orbit $M = \mathcal{O}_\lambda$, which may be identified with the homogeneous space $SO(5, \mathbb{C})/P_{\alpha_1}$ where P_{α_1} is the parabolic subgroup corresponding to the positive simple root $\{\alpha_1\}$. More specifically, we may take $\mathcal{O}_\lambda$ to be the coadjoint orbit through the element $\lambda \in \mathfrak{t} \cong \mathfrak{t}^*$ indicated in Fig. 2. The image of the equivariant 1-skeleton for the Hamiltonian T-action on $\mathcal{O}_\lambda \cong PSO(5, \mathbb{C})/P_{\alpha_1}$ is depicted in Fig. 3.

Given $S^1 \subset T$ with $M^{S^1} = M^T$ and corresponding moment map Φ_{S^1}, the level set $\Phi_{S^1}^{-1}(\eta)$ indicated (under its image under Φ) in the figure evidently lies entirely within an open Bruhat cell of M. This Bruhat cell may be modelled on a single linear T-representation with T-weights $-\alpha_2, -\alpha_1 - \alpha_2, -2\alpha_1 - \alpha_2$, which renders the explicit computation of the

Fig. 2 The type B_2 root system. The *dotted line* is the hyperplane distinguishing the positive from the negative roots. The element λ lying on the line spanned by $\alpha_1 + \alpha_2$ specifies the coadjoint orbit $\mathcal{O}_\lambda$
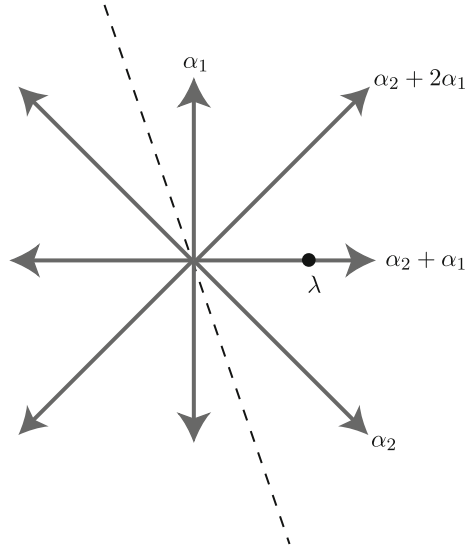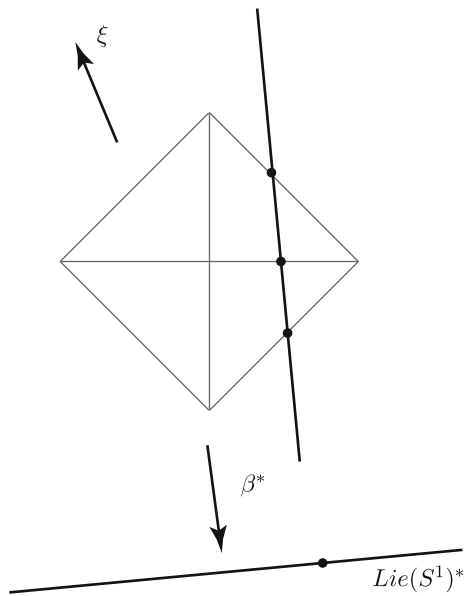

Fig. 3 In grey, we indicate the image of the equivariant 1-skeleton of $M = \mathcal{O}_\lambda \cong SO(5, \mathbb{C})/P_{\alpha_1}$. The T-fixed points are the 4 outer vertices. The *black line* intersecting the polytope represents the moment image of the level set Z of an S^1-moment map Φ_{S^1}. There are 3 critical components C_i in $\mathrm{Crit}(f)$, corresponding to the 3 *thick black dots* (the images of the C_i under Φ)

relevant finite stabilizer subgroups $\Gamma_{i,t} \subseteq S^1$ particularly straightforward. This observation motivates the discussion in the next section.

6 Semilocally Delzant spaces

We have already seen in Sects. 4 and 5 that the hypotheses of Corollary 3.2 are satisfied in several situations familiar in equivariant symplectic geometry. We will now see that the methods of proof used thus far in this manuscript allow us to make inductive use of the Main Theorem to cover more cases of orbifold symplectic quotients. Specifically, we observe that the proof

of Theorem 4.1 shows that the H-equivariant K-theory of the level set Z arising from a Delzant construction has the properties that $K_H^0(Z)$ is additive-torsion-free and $K_H^1(Z) = 0$. Therefore, for (M, ω, Φ) a Hamiltonian T-space and $\beta : H \hookrightarrow T$ a connected subtorus, if it can be shown that each of the connected components of the critical sets appearing in Theorem 1.1 can be H-equivariantly identified with a level set of a Delzant construction, then the hypotheses (3a) and (3b) of Theorem 1.1 would be satisfied, thus allowing us to apply the Main Theorem to a wider class of symplectic quotients.

To this end, we make the following definition.

Definition 6.1 Let (M, ω, Φ_H) be a Hamiltonian H-space with moment map $\Phi_H : M \to \mathfrak{h}^*$. We will say that an H-invariant subset $C \subset M$ is **semilocally Delzant with respect to H** if the following conditions are satisfied:

(1) There exists a $2n$-dimensional H-invariant symplectic submanifold $N \subseteq M$, an H-invariant open neighborhood $U \subseteq N$ of C, and a H-equivariant symplectomorphism

$$\psi : U \to V \subseteq \mathbb{C}^n$$

for an open H-invariant subset $V \subseteq \mathbb{C}^n$, where H acts linearly on $\mathbb{C}^n$, with associated moment map $\Phi_{\mathbb{C}^n} : \mathbb{C}^n \to \mathfrak{h}^*$.

(2) Under the map ψ, the set C is identified with a level set of the induced H-moment map on $\mathbb{C}^n$. In other words, $\psi(C) = \Phi_{\mathbb{C}^n}^{-1}(\eta') \subseteq \mathbb{C}^n$ for some regular value $\eta' \in \mathfrak{h}^*$.

(3) There exists $\xi \in \mathfrak{h}$ such that $\Phi_{\mathbb{C}^n}^\xi|_{\psi(C)}$ is proper and bounded below.

We take a moment to discuss situations in equivariant symplectic geometry in which we may expect the above definition to be applicable. Recall that the equivariant Darboux theorem states that, near an isolated H-fixed point $p \in M^H$, there exists an open neighborhood U_p of p which is H-equivariantly symplectomorphic to an affine space $\mathbb{C}^n$ equipped with a linear H-action (here p is identified with the origin 0 of $\mathbb{C}^n$). Under some technical assumptions (cf. [18]) which are not very restrictive in practice, it is also possible to arrange the symplectomorphism such that the H-isotypic decomposition

$$\mathbb{C}^n \cong \bigoplus_\alpha \mathbb{C}_\alpha,$$

where the sum is over weights $\alpha \in \mathfrak{h}_{\mathbb{Z}}^*$ and $\mathbb{C}_\alpha$ denotes the subspace of $\mathbb{C}^n$ of weight α, has the property that the moment map $\Phi_{\mathbb{C}^n}$ associated to this H-action has a component which is proper and bounded below. It is then evident that a closed subset C of M which lies entirely inside such an equivariant neighborhood $U_p \cong \mathbb{C}^n$ near $p \in M^T$, and which may be identified with a level set of $\Phi_{\mathbb{C}^n}$ via the equivariant Darboux theorem, is semilocally Delzant. Moreover, similar statements could be made of subsets C' of M which lie entirely in proper coordinate subspaces of $\mathbb{C}^n$ under the same equivariant identification with $U_p \subseteq \mathbb{C}^n$. Informally, we may say that H-invariant closed subsets which are "near enough to an isolated fixed point" can be semilocally Delzant as described above. In particular, this point of view leads to concrete examples of symplectic quotient constructions (e.g. of Hamiltonian H-spaces with isolated fixed points, such as those where the H-action is GKM) with critical sets C satisfying Definition 6.1.

Indeed, we note that a concrete family of examples of Hamiltonian T-spaces with well-known such equivariant neighborhoods are the flag varieties (coadjoint orbits) G/B and G/P of compact connected Lie groups. The maximal torus T of the compact connected Lie group G acts naturally on such homogeneous spaces, with fixed points corresponding to cosets

W/W_P. Moreover, G/B (similarly G/P) has a convenient open cover obtained by Weyl translates of the open Bruhat cell Bw_0B/B, where w_0 is the longest word in the Weyl group. Thus, if a closed subset C of G/B (similarly G/P) may be seen to lie entirely within the big Bruhat cell Bw_0B/B (or any of its translates), then the T-action near C may be modelled by a linear T-action on $\mathbb{C}^{\ell(w_0)}$ (here $\ell(w_0)$ denotes the Bruhat length of w_o). We illustrate a concrete example of such a situation, using a non-generic coadjoint orbit of Lie type B_3 in Example 6.4 below.

Returning to the discussion of orbifold K-theory, we first note that it is immediate from Theorem 4.1 that if C is semilocally Delzant, then $K_H^0(C)$ has no additive torsion and that $K_H^1(C) = 0$. This leads to the following.

Theorem 6.2 *Let M be a Hamiltonian T space, and let $H \subset T$ be a connected subtorus. Let $Z = \Phi_H^{-1}(\eta) \subset M$ be a level set of the H-moment map*

$$\Phi_H : M \to \mathfrak{h}^*$$

and $\mathfrak{X} = [Z/H]$ be the orbifold obtained as a symplectic quotient $M /\!/ H$. Let $\xi \in \mathfrak{t}$ be such that $T^\xi = T$, and $f = \Phi^\xi|_Z$ the corresponding moment map restricted to Z. Suppose that

(1) *f is proper and bounded below,*
(2) *for all $t \in H$, the set of connected components $\pi_0(\mathrm{Crit}(f|_{Z^t}))$ is finite,*
(3) *for all $t \in H$, each connected component C of $\mathrm{Crit}(f|_{Z^t})$ is semi-locally Delzant with respect to H.*

Then $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X})$ has no additive torsion. Furthermore, $K_H^1(Z^t) = 0$ for all $t \in H$.

Proof By Theorem 4.1, $\mathbb{K}_{\mathrm{orb}}([C/H])$ has no additive torsion for each connected component C of $Crit(f|_{Z^t})$ for all $t \in H$. In particular, $K_H^0(C)$ has no torsion. Since we also have $K_H^1(C) = 0$ for all critical sets, we have satisfied the hypotheses of Theorem 1.1. Hence $\mathbb{K}_{\mathrm{orb}}(\mathfrak{X})$ has no additive torsion. □

Remark 6.3 We note that if the level set Z itself is semilocally Delzant, then by transferring all analysis to the appropriate equivariant Darboux neighborhood $U \subseteq \mathbb{C}^n$ and using the same argument as in Sect. 4, we immediately see that for all $t \in H$, all connected components C of $\mathrm{Crit}(f|_{Z^t})$ are semilocally Delzant with respect to H. Hence, in this case the hypothesis (3) above is automatically satisfied.

Example 6.4 We close with an example of a symplectic quotient of a type B_3 coadjoint orbit by a 2-dimensional torus. Since the subtorus is dimension 2, Theorem 5.1 does not apply, but we may use Theorem 6.2. Recall that the complex form of the compact Lie group of type B_3 is $PSO(7, \mathbb{C})$. The maximal torus T is 3-dimensional, and the root system is depicted in Fig. 4. We denote the associated moment map by Φ.

We choose to work with a non-generic coadjoint orbit $\mathcal{O}_\lambda$ which may be identified with the complex homogeneous space $PSO(7, \mathbb{C})/P_{\alpha_2,\alpha_3}$ where P_{α_2,α_3} is the parabolic subgroup corresponding to the subset of the positive simple roots $\{\alpha_2, \alpha_3\}$. We choose λ lying on the positive span of the positive root $L_1 = \alpha_1 + \alpha_2 + \alpha_3$ as in Fig. 4. The GKM graph of $\mathcal{O}_\lambda$ is also schematically shown. The image of the equivariant 1-skeleton of $M = \mathcal{O}_\lambda$ includes the three 2-dimensional interior quadrilaterals given by the convex hull of the roots $\{\pm L_1, \pm L_2\}, \{\pm L_2, \pm L_3\}, \{\pm L_1, \pm L_3\}$ respectively.

Let $T' \subset T$ be the 2-dimensional connected subtorus of T corresponding to the 2-plane spanned by the roots $\{\pm L_1, \pm L_2\}$, with corresponding projection $\pi_{T'} : \mathfrak{t}^* \to \mathrm{Lie}(T')^*$. We wish to compute $\mathbb{K}_{\mathrm{orb}}$ of the symplectic quotient $\mathcal{O}_\lambda /\!/ T'$. The preimage $\pi_{T'}^{-1}(\eta) \cap \Delta$ in

Fig. 4 The root diagram for type B_3 with positive simple roots $\alpha_1, \alpha_2, \alpha_3$ (for details, see [14, §19.3]). The element $\lambda \in \mathfrak{t}^*$ lies on the positive span of the positive root $L_1 = \alpha_1 + \alpha_2 + \alpha_3$
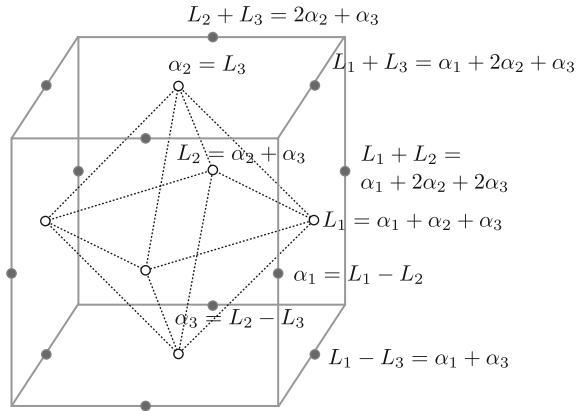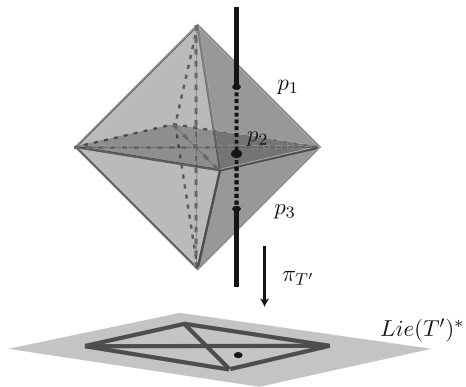

$$L_2 + L_3 = 2\alpha_2 + \alpha_3$$
$$\alpha_2 = L_3 \qquad L_1 + L_3 = \alpha_1 + 2\alpha_2 + \alpha_3$$
$$L_2 = \alpha_2 + \alpha_3 \qquad L_1 + L_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3$$
$$L_1 = \alpha_1 + \alpha_2 + \alpha_3$$
$$\alpha_1 = L_1 - L_2$$
$$\alpha_3 = L_2 - L_3$$
$$L_1 - L_3 = \alpha_1 + \alpha_3$$

Fig. 5 The GKM graph for $M = \mathcal{O}_\lambda \cong PSO(7, \mathbb{C})/P_{\alpha_2,\alpha_3}$. The *thick line* and *thick black dots* schematically illustrate the (image under Φ of the) inverse images $Z := (\Phi_{T'})^{-1}(\eta)$ and the critical components of $\mathrm{Crit}(\Phi^\xi|_Z)$, respectively

$\Delta = \Phi(\mathcal{O}_\lambda)$ of a generic regular value $\eta \in Lie(T')^*$ is depicted as the thick interval in Fig. 5.

We wish now to show that the full orbifold K-theory of the quotient $\mathcal{O}_\lambda /\!/ T'$ is free of additive torsion by using Theorem 6.2. There are several ways to proceed. The first method, which depends on Remark 6.3, is to simply observe that the full level set Z is semilocally Delzant, owing to the fact that it lies entirely in the single open Bruhat cell U_p centered at the T-fixed point p corresponding to the root $L_1 = \alpha_1 + \alpha_2 + \alpha_3$, and the T-action and corresponding moment map Φ restricted to this Bruhat cell may be identified with that of a linear T-action on $\mathbb{C}^5$ with weights $\{-L_1, -L_1 \pm L_2, -L_1 \pm L_3\}$ on the coordinates. The T'-action is the restriction of this linear T-action, hence Z is semilocally Delzant with respect to T'. By Remark 6.3 we may immediately apply Theorem 6.2, as desired.

In order to illustrate the concrete, straightforward nature of our method of computation, for this example we also briefly sketch the explicit analysis of each of the components of $\mathrm{Crit}(f)$ for appropriate $f = \Phi^\xi|_Z$. Analysis of $\mathrm{Crit}(f|_{Z^t})$, for $t \neq 1$, would of course be similar. We begin by choosing ξ generic such that $\mathrm{Crit}(f)$ consists of the three components schematically indicated in Fig. 5.

Observe that the situations of the two exterior points p_1, p_3 in $\pi_{T'}^{-1}(\eta) \cap \Delta$ lying on the boundary $\partial\Delta$ are evidently symmetric, so it suffices to do the computations for only one of them. We begin with the top exterior point p_1. A straightforward analysis of the linear T-action on the Bruhat cell described above shows that $\Phi^{-1}(p_1) \subseteq \mathcal{O}_\lambda$ consists of a sin-

gle T-orbit diffeomorphic to T'. Moreover, the intersection of the stabilizer of the Bruhat cell with T' is trivial, so p_1 corresponds to a free T'-orbit. Hence the contribution to the full orbifold K-theory coming from p_1 is the (ordinary) K-theory of a point, and is hence torsion-free.

We now proceed with the interior point p_2. (One way to view this computation is to recall that the horizontal quadrilateral obtained as the convex hull of the roots $\{\pm L_1, \pm L_2\}$ corresponds to a subvariety of $PSO(7, \mathbb{C})/P_{\alpha_2,\alpha_3}$ which may be identified with the homogeneous space of $PSO(5, \mathbb{C})$ of type B_2 studied in a previous example, although this is not necessary for the computation.) Another straightforward analysis of linear T-actions, using the explicit list of T-weights given above, yields that the corresponding symplectic quotient is the "teardrop" orbifold, i.e. the weighted projective space $\mathbb{P}(1, 2)$ (following notation of [15]). Hence the contribution to the full orbifold K-theory of $\mathcal{O}_\lambda /\!/_\mu T'$ coming from the interior point p_2 is that associated to $\mathbb{P}(1, 2)$, which is explicitly computed in [15], and has no additive torsion.

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