

## ORBIFOLD COHOMOLOGY OF HYPERTORIC VARIETIES

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Hypertoric varieties are hyperkähler analogues of toric varieties, and are constructed as abelian hyperkähler quotients  $T^*\mathbb{C}^n // T$  of a quaternionic affine space. Just as symplectic toric orbifolds are determined by labelled polytopes, orbifold hypertoric varieties are intimately related to the combinatorics of hyperplane arrangements. By developing hyperkähler analogues of symplectic techniques developed by Goldin, Holm, and Knutson, we give an explicit combinatorial description of the Chen–Ruan orbifold cohomology of an orbifold hypertoric variety in terms of the combinatorial data of a rational cooriented weighted hyperplane arrangement  $\mathcal{H}$ . We detail several explicit examples, including some computations of orbifold Betti numbers (and Euler characteristics).

*Keywords:* Hyperkähler quotients; hypertoric varieties; orbifolds; orbifold cohomology; hyperplane arrangements.

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### 1. Introduction

The main result of this manuscript is an explicit combinatorial computation of the Chen–Ruan orbifold cohomology  $H_{CR}^*(M)$  [3] of an orbifold hypertoric variety  $M$ . Hypertoric varieties are hyperkähler analogues of toric varieties, and were first introduced by Bielawski and Dancer [1], and further studied by Konno [10, 11] and Hausel and Sturmfels [7], among others. Just as a symplectic toric orbifold is determined by a labelled polytope, the theory of orbifold hypertoric varieties is intimately related to the combinatorial data of a related rational cooriented hyperplane arrangement  $\mathcal{H}$ . Our description of  $H_{CR}^*(M)$  is given purely in terms of this arrangement  $\mathcal{H}$ . The fact that these hypertoric varieties are constructed as

hyperkähler quotients  $T^*\mathbb{C}^n // T$  of a quaternionic affine space  $T^*\mathbb{C}^n \cong \mathbb{H}^n$  (via the hyperkähler analogue of the Delzant construction of Kähler toric varieties) is crucial to our techniques.

Hyperkähler quotients appear in many areas of mathematics. For instance, in representation theory, Nakajima’s quiver varieties give rise to geometric models of representations (see e.g. [13–15]). Furthermore, many moduli spaces appearing in physics, such as spaces of Yang–Mills instantons on 4-manifolds or the solutions to the Yang–Mills–Higgs equations on a Riemann surface, arise via hyperkähler quotient constructions. In each case, the study of topological invariants, such as cohomology rings or  $K$ -theory, of these quotients are of interest. In the case of hypertoric varieties, there are also close connections between the (ordinary or Borel-equivariant) cohomology rings of the varieties and the combinatorial theory of the corresponding hyperplane arrangements [6, 7, 10, 11]. Generalizing known such results to the orbifold case is of current interest. For example, recent work of Proudfoot and Webster [16, Sec. 6] on the intersection cohomology of singular hypertoric varieties and the cohomology of their orbifold resolutions contains cohomological formulas which only apply in the unimodular case; it would be of interest to know whether there are orbifold versions of their statements.

In this paper, we focus on the combinatorics of the hyperplane arrangement associated to the **Chen–Ruan orbifold cohomology** of orbifold hypertoric varieties. Chen–Ruan orbifold cohomology rings were introduced in [3] as the degree 0 piece of the Gromov–Witten theory of an orbifold, following work in physics [18]. This ring carries, in addition to the data of the usual singular cohomology ring of the underlying space, more delicate information (e.g. about the orbifold structure groups). Additively,  $H_{CR}^*(M)$  is simply the usual singular cohomology of the inertia orbifold  $\widetilde{M}$  associated to  $M$ ; the product structure, on the other hand, is much more subtle, incorporating the data of higher twisted sectors. In this manuscript, we provide an explicit presentation, via generators and relations, of this Chen–Ruan cohomology ring for a class of orbifold hypertoric varieties.

Our approach is to develop hyperkähler analogues of the symplectic-geometric techniques as introduced by Goldin, Holm, and Knutson in [5] to compute Chen–Ruan orbifold cohomology. As in their work, we take advantage of the fact that a hypertoric variety is by construction a global quotient of a manifold by a torus. We now briefly recall the main results of [5]. Let  $T$  be a compact connected torus, let  $N$  be a compact Hamiltonian  $T$ -manifold moment map  $\mu : N \rightarrow \mathfrak{t}^*$ , and suppose that  $\alpha$  is a regular value of  $\mu$ . Then the inclusion  $\mu^{-1}(\alpha) \hookrightarrow N$  induces a natural ring<sup>a</sup> homomorphism, often called the **Kirwan map**:

$$\kappa : H_T^*(N) \twoheadrightarrow H_T^*(\mu^{-1}(\alpha)) \cong H^*(N//_\alpha T), \tag{1.1}$$

which is a surjection [9]. Here  $N//_\alpha T := \mu^{-1}(\alpha)/T$  is by definition the symplectic quotient of  $N$  at  $\alpha$ . The main result of [5] is an orbifold cohomology version of (1.1)

<sup>a</sup>In this paper, we take *rational* coefficients for all cohomology rings.

for abelian symplectic quotients. In other words, they show that the inclusion  $\mu^{-1}(\alpha) \rightarrow N$  induces a surjective ring map (the “**orbifold Kirwan map**”)

$$\kappa_{NH} : NH_T^{*,\diamond}(N) \twoheadrightarrow NH_T^{*,\diamond}(\mu^{-1}(\alpha)) \cong H_{CR}^*(M), \tag{1.2}$$

where the domain is a new ring which they define: it is the **inertial cohomology ring**  $NH_T^{*,\diamond}(N)$  of the  $T$ -space  $N$ . (Here it is nontrivial that  $\kappa_{NH}$  is a ring homomorphism; the same subtlety also arises in the hyperkähler case.) Moreover, they give an explicit description of the kernel of  $\kappa_{NH}$ . Their proof relies on symplectic-geometric properties of the fixed point sets  $N^t$  for  $t \in T$  in abelian Hamiltonian spaces, as well as on the original Kirwan surjectivity result (1.1).

In this paper, we prove a parallel story in the hypertoric setting. A direct hyperkähler analogue of (1.2) is nontrivial for several reasons, foremost among which is that it is not known whether the hyperkähler analogue of (1.1) is, in general, surjective. However, in the special case where  $M$  is a smooth or orbifold hypertoric variety [7, 11], obtained as a hyperkähler quotient of  $T^*\mathbb{C}^n$ , a hyperkähler analogue of (1.1) does hold, allowing us to obtain results in this setting. The other nontrivial issues are the non-compactness of the hypertoric varieties (in [5], all orbifolds are assumed compact) and the analysis of the hyperkähler-geometric properties of the fixed point sets  $N^t \subseteq N$  for  $t \in T$ . We deal with these issues in Sec. 4 to obtain the following. Let  $\mu_{HK} : M \rightarrow \mathfrak{t}^* \oplus \mathfrak{t}_{\mathbb{C}}^*$  denote the hyperkähler moment map on  $T^*\mathbb{C}^n$ , and  $T^*\mathbb{C}^n \mathbin{////} T$  its hyperkähler quotient at a regular value  $(\alpha, \alpha_{\mathbb{C}}) \in \mathfrak{t}^* \oplus \mathfrak{t}_{\mathbb{C}}^*$  as described in Sec. 2.

**Theorem 1.1.** *Let  $M$  be an orbifold hypertoric variety  $T^*\mathbb{C}^n \mathbin{////} T$ . There is a surjective ring homomorphism*

$$\kappa_{NH}^{\Gamma} : NH_T^{*,\Gamma}(T^*\mathbb{C}^n) \twoheadrightarrow H_{CR}^*(M), \tag{1.3}$$

where  $\Gamma$  is the subgroup of  $T$  generated by finite stabilizers,  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$  is the  $\Gamma$ -subring of the inertial cohomology ring  $NH_T^{*,\diamond}(T^*\mathbb{C}^n)$ , and  $H_{CR}^*(M)$  is the Chen–Ruan cohomology of  $M$ .

The point of Theorem 1.1 is that we can, in principle, compute the orbifold cohomology of the hypertoric variety  $M = T^*\mathbb{C}^n \mathbin{////} T$  as a quotient of  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$  by the kernel of (1.3). In the spirit of [10, 11, 7, 6], we give an explicit algorithm for computing both the domain  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$  and the ideal  $\ker(\kappa_{NH}^{\Gamma})$  in terms of the combinatorics of a central rational cooriented weighted hyperplane arrangement  $\mathcal{H}_{\text{cent}}$  along with a choice of simple affinization  $\mathcal{H}$ . This combinatorial data is obtained from the data of the  $T$ -action on  $T^*\mathbb{C}^n$  and an appropriate choice of level set of the hyperkähler moment map (explained in detail in Sec. 2). We now give a rough statement of our main theorem, which gives a flavor of the ingredients in the computation; the precise version is Theorem 5.1.

**Theorem 1.2.** *Let  $M = T^*\mathbb{C}^n \mathbin{////} T$  be an orbifold hypertoric variety. Let  $\mathcal{H} = \{H_i\}_{i=1}^n$  be a simple affine rational cooriented hyperplane arrangement with positive normal vectors  $\{a_i\}_{i=1}^n$  associated to  $M$  as described in Sec. 2. Then the Chen–Ruan*

cohomology of  $M$  is given by

$$H_{CR}^*(M) \cong \mathbb{Q}[u_1, u_2, \dots, u_n][\{\gamma_t\}_{t \in \Gamma}] / \mathcal{I} + \mathcal{J} + \mathcal{K} + \langle \gamma_{\text{id}} - 1 \rangle,$$

where

- $\Gamma$  is a finite subgroup of  $T$  determined by linear independence relations among the  $\{a_i\}_{i=1}^n$ , made precise in (5.4).
- $\mathcal{I}$  is an ideal determined by  $T$ -weight data coming from the action of  $T$  on  $T^*\mathbb{C}^n$  specified by  $\mathcal{H}$ , made precise in Proposition 5.2.
- $\mathcal{J}$  is an ideal generated by linear relations coming from an exact sequence of Lie algebras  $0 \rightarrow \mathfrak{t} \rightarrow \mathfrak{t}^n \rightarrow \mathfrak{t}^d \rightarrow 0$  given by the  $T$ -action on  $T^*\mathbb{C}^n$  specified by  $\mathcal{H}$ , made precise in (2.1).
- $\mathcal{K}$  is determined by intersection data of the hyperplanes  $H_i$  in  $\mathcal{H}$ , and made precise in Proposition 5.3.

In summary, this manuscript can be viewed in any of the following ways. First, it is an example of an explicit computation of the Chen–Ruan orbifold cohomology of hyperkähler quotients, and a further development, in the hyperkähler setting, of the definition and use of inertial cohomology as introduced in [5]. In particular, we note that our methods would also apply to any class of hyperkähler quotients for which there exists an appropriate analogue of the Kirwan surjection (1.1). Similarly, although in this manuscript we restrict our attention to  $\mathbb{Q}$  coefficients for our cohomology rings, if a  $\mathbb{Z}$ -coefficient analogue of the Kirwan surjection for orbifold hypertoric varieties is proven, then our methods will easily generalize to the setting of  $\mathbb{Z}$  coefficients. Second, it is another exploration of the relationship between the geometry of hypertoric varieties and the combinatorics of hyperplane arrangements. Finally, it is the hyperkähler-geometric analogue of the algebraic-geometric description of the Chow ring of toric Deligne–Mumford stacks in [2].

In [8], Jiang and Tseng independently develop techniques for an algebraic-geometric version of these results by defining “hypertoric DM stacks” using extended stacky fans, following work of [2]. Their work applies to the sub-class of hypertoric varieties  $M$  obtained by hyperkähler quotients at regular values of the form  $(\alpha, 0)$ . In this case, there is a simple affine hyperplane arrangement  $\mathcal{H}$  in  $\mathfrak{t}^*$  determined by the data of a moment map for a residual torus action on  $M$ ; the results of [8] are phrased in terms of this arrangement  $\mathcal{H}$ . Our results, on the other hand, apply to an orbifold hypertoric variety obtained as a quotient at any regular value  $(\alpha, \alpha_{\mathbb{C}})$ . This is because we do not keep track of the hyperkähler structure of the quotient (which does depend on this choice of level set); the Chen–Ruan orbifold cohomology of the quotient turns out to be *independent* of this choice, i.e. is the same for any regular value. The main difference between the approach taken in this manuscript and [8] is that Jiang and Tseng begin with the data of a simple hyperplane arrangement  $\mathcal{H}$  and then directly construct the hypertoric DM stack associated to  $\mathcal{H}$ , which has coarse moduli space the corresponding orbifold hypertoric variety. As a result, they compute the product in the orbifold Chow ring entirely in terms of the quotient hypertoric variety. In contrast, our method is to

work almost entirely “upstairs” on  $T^*\mathbb{C}^n$  with a linear  $T$ -action, before taking a hyperkähler quotient. This simplifies some computations (as in [5]) by allowing us to work with linear  $T$ -representations, and carries the information of a family of hypertoric varieties at once.

Since orbifold Chen–Ruan cohomology reduces to ordinary cohomology when  $M$  is smooth, both our work and that of [8] reduce to the description of  $H^*(M)$  given in [11] (see also [7]) in the case when  $M$  is a smooth hypertoric variety. As Jiang and Tseng illustrate [8], this can be useful to show that the ordinary cohomology of a smooth hyperkähler crepant resolution of  $\mathbb{C}^2/\mathbb{Z}_n$  as constructed by Kronheimer [12] is isomorphic to the orbifold cohomology of  $\mathbb{C}^2/\mathbb{Z}_n$ , which can be computed using [4].

We now give a summary of the contents of the paper. In Sec. 2, we give a brief account of the construction of hypertoric varieties as a hyperkähler quotient, based on the data of a hyperplane arrangement. In Sec. 3, we briefly recall the definition of inertial cohomology given in [5]. Then in Sec. 4, we prove that there exists a surjection in inertial cohomology as in (1.2). We give a combinatorial description of the Chen–Ruan orbifold cohomology of a hypertoric variety, based on the data of the hyperplane arrangement  $\mathcal{H}$ , in Sec. 5. In Sec. 6, we work out in detail several explicit examples, including some computations of orbifold Betti numbers and orbifold Euler characteristics. The Appendix contains a detailed discussion of the isomorphism between inertial cohomology of a  $T$ -space  $Z$  and the Chen–Ruan cohomology of the quotient  $X = Z/T$  (also discussed for the compact case in [5]), as well as a careful proof of the correspondence between Chen and Ruan’s definition of the obstruction bundle with that used in the algebraic geometry literature (e.g. [2, 4]).

## 2. Background: Hypertoric Varieties

We first briefly describe the construction of hypertoric varieties in order to set the notation and conventions to be used throughout the rest of the paper. We refer the reader to [1, 6, 7] for a more leisurely account.

We begin with the hyperkähler space  $\mathbb{H}^n$ , thought of as a holomorphic cotangent bundle  $T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$ . This is a hyperkähler manifold with real symplectic form  $\omega_{\mathbb{R}}$  given by the identification with  $T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$  and  $\omega_{\mathbb{C}}$  the canonical holomorphic symplectic form on a cotangent bundle. The standard linear diagonal action of the compact torus  $T^n$  on  $\mathbb{C}^n$  induces an action on the holomorphic cotangent bundle  $T^*\mathbb{C}^n$  which is hyperhamiltonian [1]. We will refer to this action as the **standard hyperhamiltonian action of  $T^n$  on  $T^*\mathbb{C}^n$** . The hyperkähler  $T^n$ -moment map  $\tilde{\mu}_{HK} = (\tilde{\mu}_{\mathbb{R}}, \tilde{\mu}_{\mathbb{C}})$  on  $T^*\mathbb{C}^n$  is given as follows. Let  $\{u_i\}_{i=1}^n$  be a dual basis to  $\{\varepsilon_i\}_{i=1}^n$  in  $(\mathfrak{t}^n)^*$ , and let  $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_n) \in T^*\mathbb{C}^n$ , where the  $z_i$  are the base variables and the  $w_i$  are the fiber variables. We have

$$\tilde{\mu}_{\mathbb{R}}(z, w) = \frac{1}{2} \sum_{i=1}^n (\|z_i\|^2 - \|w_i\|^2) u_i \in (\mathfrak{t}^n)^*, \quad \text{and}$$

$$\tilde{\mu}_{\mathbb{C}}(z, w) = \sum_{i=1}^n z_i w_i u_i \in (\mathfrak{t}_{\mathbb{C}}^n)^*.$$

Let  $\mathcal{H}_{\text{cent}} = \{H_i^{\text{cent}}\}_{i=1}^n$  be a central rational cooriented weighted hyperplane arrangement in  $(\mathfrak{t}^d)^*$  with positive normal vectors  $\{a_i\}_{i=1}^n$  in  $\mathfrak{t}_{\mathbb{Z}}^d$ . Here, “weighted” means that we do not require the  $a_i$  to be primitive vectors. We now use this data to restrict the  $T^n$  action to that of a subtorus. Let  $\{\varepsilon_i\}_{i=1}^n$  be a basis of  $\mathfrak{t}^n$ . Define a linear map  $\beta : \mathfrak{t}^n \rightarrow \mathfrak{t}^d$  by  $\beta(\varepsilon_i) = a_i \in \mathfrak{t}_{\mathbb{Z}}^d$ . Let  $\mathfrak{t} = \mathfrak{t}^k := \ker(\beta) \subseteq \mathfrak{t}^n$ , where  $k = n - d$ , with inclusion  $\iota : \mathfrak{t}^k \hookrightarrow \mathfrak{t}^n$ . This yields an exact sequence

$$0 \longrightarrow \mathfrak{t} = \mathfrak{t}^k \xrightarrow{\iota} \mathfrak{t}^n \xrightarrow{\beta} \mathfrak{t}^d \longrightarrow 0, \tag{2.1}$$

$$\varepsilon_i \longmapsto a_i$$

which on the one hand exponentiates to an exact sequence

$$1 \longrightarrow T = T^k \xrightarrow{\exp \iota} T^n \xrightarrow{\exp \beta} T^d \longrightarrow 1, \tag{2.2}$$

and on the other hand dualizes to the exact sequence

$$0 \longrightarrow (\mathfrak{t}^d)^* \xrightarrow{\beta^*} (\mathfrak{t}^n)^* \xrightarrow{\iota^*} \mathfrak{t}^* = (\mathfrak{t}^k)^* \longrightarrow 0. \tag{2.3}$$

$$u_i \longmapsto \lambda_i := \iota^* u_i$$

We will always assume that the set of integer vectors  $\{a_i\}_{i=1}^n$  spans  $\mathfrak{t}^d$  over  $\mathbb{Z}$ , so that the kernel  $T = T^k := \ker(\exp \beta)$  is connected; this assumption is also made in [7].

Now we restrict the  $T^n$ -action on  $T^*\mathbb{C}^n$  to the subtorus  $T$ . Let  $\lambda_i := \iota^* u_i \in (\mathfrak{t}^k)^*_{\mathbb{Z}}$  as in (2.3). Then  $\lambda_i$  is the  $T$ -weight defining the action of the subtorus  $T$  on the  $i$ th coordinate of  $\mathbb{C}^n$ . Let  $\exp \lambda_i$  denote the corresponding element in  $\text{Hom}(T, S^1)$ . Since the action of  $T$  on  $T^*\mathbb{C}^n$  is given by the natural lift of that on  $\mathbb{C}^n$ , we have that for  $t \in T, (z, w) \in T^*\mathbb{C}^n$ ,

$$t \cdot (z, w) = ((\exp \lambda_1)(t)z_1, \dots, (\exp \lambda_n)(t)z_n, (\exp \lambda_1)(t)^{-1}w_1, \dots, (\exp \lambda_n)(t)^{-1}w_n). \tag{2.4}$$

The moment maps for the hyperhamiltonian  $T$ -action on  $T^*\mathbb{C}^n$  are given by composing  $\tilde{\mu}_{HK}$  with the linear projection  $\iota^* : (\mathfrak{t}^n)^* \rightarrow \mathfrak{t}^*$ . Thus we obtain the formulas

$$\mu_{\mathbb{R}}(z, w) = \frac{1}{2} \sum_{i=1}^n (\|z_i\|^2 - \|w_i\|^2) \lambda_i \in \mathfrak{t}^*, \quad \text{and} \quad \mu_{\mathbb{C}}(z, w) = \sum_{i=1}^n z_i w_i \lambda_i \in \mathfrak{t}_{\mathbb{C}}^*. \tag{2.5}$$

We will assume throughout that  $\lambda_i \neq 0, \forall i$ .

In order to specify the hyperkähler quotient, we now pick a regular value  $(\alpha, \alpha_{\mathbb{C}}) \in \mathfrak{t}^* \oplus \mathfrak{t}_{\mathbb{C}}^* \cong (\mathfrak{t}^*)^3$  at which to reduce. Any element  $\alpha \in \mathfrak{t}^*$  specifies an affinization  $\mathcal{H} = \{H_i\}_{i=1}^n$  of  $\mathcal{H}_{\text{cent}}$  via the equations

$$H_i := \{x \in (\mathfrak{t}^d)^* : \langle x, a_i \rangle = \langle -\tilde{\alpha}, \varepsilon_i \rangle\},$$

where  $\langle \cdot, \cdot \rangle$  denotes the natural pairing of a vector space and its dual, and  $\tilde{\alpha} \in (\mathfrak{t}^n)^*$  is a lift of  $\alpha$ , i.e.  $\iota^* \tilde{\alpha} = \alpha$ . (A different choice of lift just translates the whole hyperplane arrangement by a constant.)

In particular, the choice of parameter  $(\alpha, \alpha_{\mathbb{C}}) \in (\mathfrak{t}^*)^3$  corresponds to *three* separate choices of affinization of the central arrangement  $\mathcal{H}_{\text{cent}}$ . Hence a hypertoric variety is determined by the combinatorial data of a central weighted arrangement  $\mathcal{H}_{\text{cent}}$  and, additionally, 3 choices of affinization of  $\mathcal{H}_{\text{cent}}$ . However, in the case of our computation, this data can be simplified considerably. This is because a preimage  $\mu_{HK}^{-1}(\alpha, \alpha_{\mathbb{C}})$  of a regular value of  $\mu_{HK}$  is  $T^n$ -equivariantly diffeomorphic to the preimage  $\mu_{HK}^{-1}(\alpha', \alpha'_{\mathbb{C}})$  of any other regular value; this can be seen by an argument essentially equivalent to the proof of [6, Lemma 2.1]. Hence the Chen–Ruan cohomology of  $T^*\mathbb{C}^n \mathbin{////}_{(\alpha, \alpha_{\mathbb{C}})} T$  can be seen to be isomorphic to that of  $T^*\mathbb{C}^n \mathbin{////}_{(\alpha', \alpha'_{\mathbb{C}})} T$ , as will be discussed further in Remark 3.3. In other words, the Chen–Ruan cohomology of an orbifold hypertoric variety is determined by the original central arrangement  $\mathcal{H}_{\text{cent}}$ , and is independent of these choices of affine structures given by regular values. In practice, however, it is useful to pick a convenient affinization with which to work. Namely, if  $\alpha \in \mathfrak{t}^*$  is chosen such that the corresponding affinization of  $\mathcal{H}_{\text{cent}}$  is **simple**<sup>b</sup>, then  $(\alpha, 0) \in \mathfrak{t}^* \oplus \mathfrak{t}_{\mathbb{C}}^*$  is a regular value [1, Theorem 3.3]. Here, since the last two parameters are both 0, only the first factor gives rise to a nontrivial affinization of  $\mathcal{H}_{\text{cent}}$ . We will denote by  $\mathcal{H} = \{H_i\}_{i=1}^n$  this *simple* affine rational cooriented weighted hyperplane arrangement obtained from the data of  $\mathcal{H}_{\text{cent}}$  and an appropriate  $\alpha \in \mathfrak{t}^*$ .

### 3. Background: Inertial Cohomology

We begin with a brief account of inertial cohomology as developed in [5], which gives us a model for computing the orbifold cohomology of the hyperkähler quotients constructed in Sec. 2. Readers already familiar with the definition of orbifold cohomology in the sense of Chen and Ruan will find Sec. 3.1 straightforward, since the product on the inertial cohomology of a  $T$ -space  $Z$  is defined precisely to mimic the Chen–Ruan product in the case that the quotient  $X = Z/T$  is an orbifold, where  $T$  acts with finite stabilizers on  $Z$ . The contribution of [5] is to notice that in other cases of  $T$ -spaces (such as Hamiltonian  $T$ -spaces), the product on inertial cohomology can be described in terms of a (different) product defined in terms of fixed point data. This (different) product, which is easier to compute, is briefly recalled in Sec. 3.2; it will play a key role in our computation of the Chen–Ruan orbifold cohomology of hypertoric varieties.

#### 3.1. Inertial cohomology and the $\smile$ product

Let  $N$  be a stably complex  $T$ -space. For any  $t \in T$ , let  $N^t$  denote the  $t$ -fixed points. Since  $T$  is abelian, each  $N^t$  is also a  $T$ -space.

<sup>b</sup>A hyperplane arrangement is simple if any subset of  $\ell$  hyperplanes intersect in codimension  $\ell$ .

**Definition 3.1.** The **inertial cohomology** of the space  $N$  is, as a  $H_T^*(pt)$ -module, given by

$$NH_T^{*\diamond}(N) := \bigoplus_{t \in T} H_T^*(N^t), \tag{3.1}$$

where the sum indicates the  $\diamond$  grading, i.e.  $NH_T^{*,t}(N) := H_T^*(N^t)$ .

The  $*$  grading on the left-hand side is a *real*-valued grading defined in [17] which is obtained from the  $*$  grading on the right-hand side by a shift depending (in this case) on the  $T$ -action; see [5, Sec. 3] for a detailed discussion.

**Remark 3.1.** In the case of the orbifold hypertoric varieties under consideration in this paper, it will turn out that the  $*$  grading is integral and always even.

Although the stably complex structure does not enter into the definition of the inertial cohomology as an additive group, it is an essential ingredient in the definition of its product structure, which we now discuss; first, however, we warn the reader that the definition of the product on  $NH_T^{*\diamond}(N)$ , which we denote by  $a \smile b$ , is *not* necessary for understanding the statement of our main Theorem 1.1, but is necessary for the proof. We include a brief definition only for completeness, and refer the reader to [5, Sec. 3] for details.

To describe the product, we make use of the top Chern class of the “obstruction bundle”, which is a vector bundle over connected components of certain submanifolds of  $N$ . More specifically, let  $t_1, t_2 \in T$ , let  $H = \langle t_1, t_2 \rangle$  be the subgroup they generate, and  $N^H$  the submanifold of points fixed by  $H$ . For any connected component  $Y$  of  $N^H$ , the normal bundle  $\nu(Y, N)$  of  $Y$  in  $N$  is naturally equipped with an  $H$ -action. We may decompose  $\nu(Y, N)$  into isotypic components with respect to the  $H$ -action:

$$\nu(Y, N) = \bigoplus_{\lambda \in \hat{H}} I_\lambda,$$

where  $\hat{H}$  denotes the character group of  $H$ .

**Definition 3.2.** Let  $\lambda \in \hat{H}$  and  $t \in H$ . For any connected component  $Y$  of  $N^H$ , we define the **logweight of  $t$  with respect to  $\lambda$** , denoted  $a_\lambda(t)$ , to be the real number in  $[0, 1)$  such that  $\lambda(t) = e^{2\pi i a_\lambda(t)}$ .

Note that for any elements  $t_1, t_2 \in H$  and for any connected component of  $N^H$ , the sum  $a_\lambda(t_1) + a_\lambda(t_2) + a_\lambda((t_1 t_2)^{-1})$  must be 0, 1, or 2.

**Definition 3.3.** The **obstruction bundle** is a vector bundle over each component  $Y$  of  $N^H$  specified by

$$E|_Y := \bigoplus_{\substack{\lambda \in \hat{H} \\ a_\lambda(t_1) + a_\lambda(t_2) + a_\lambda((t_1 t_2)^{-1}) = 2}} I_\lambda,$$

where  $\nu(Y, N) = \bigoplus I_\lambda$ . We write  $E \rightarrow N^H$  to denote the union over all connected components. Note that the dimension may vary over components. The **virtual**



**fundamental class**  $\varepsilon \in H_T^*(N^H)$  is given by

$$\varepsilon := \sum_{Y \in \pi_0(N^H)} e(E|_Y),$$

where  $e(E|_Y)$  is the  $T$ -equivariant Euler class of  $E|_Y$ , considered as an element of  $H_T^*(Y)$ .

Now let  $e_i : N^H \rightarrow N^{t_i}$  for  $i = 1, 2$ ; and  $\bar{e}_3 : N^H \rightarrow N^{t_1 t_2}$  denote the natural inclusions. These induce pullbacks  $e_i^* : H_T^*(N^{t_i}) \rightarrow H_T^*(N^H)$  for  $i = 1, 2$ ; and the pushforward  $(\bar{e}_3)_* : H_T^*(N^H) \rightarrow H_T^*(N^{t_1 t_2})$ . Let  $a \in NH_T^{*,t_1}(N)$  and  $b \in NH_T^{*,t_2}(N)$  be homogeneous classes in  $\diamond$ . Then we define the product  $a \smile b \in NH_T^{*,t_1 t_2}(N)$ , a homogeneous class in  $\diamond$ , to be

$$a \smile b := (\bar{e}_3)_*(e_1^*(a) \cdot e_2^*(b) \cdot \varepsilon) \in H_T^*(N^{t_1 t_2}) = NH_T^{*,t_1 t_2}(N) \subset NH_T^{*,\diamond}(N),$$

where the product  $\cdot$  on the right-hand side is the usual product on  $H_T^*(N^H)$ . Extending linearly, the product is defined on any two classes  $a, b$ .

**Remark 3.2.** It follows immediately from the definition of the product that, for any subgroup  $\Gamma$  of  $T$ , there is a subring  $NH_T^{*,\Gamma}(N)$  given by

$$NH_T^{*,\Gamma}(N) = \bigoplus_{t \in \Gamma} H_T^*(N^t).$$

We call this ring the  $\Gamma$ -subring of  $NH_T^{*,\diamond}(N)$ .

**Remark 3.3.** It is straightforward to show from the definition of  $NH_T^{*,\diamond}$  that if  $Z, Z'$  are stably complex  $T$ -spaces equipped with locally free  $T$ -actions and there exists a  $T$ -equivariant diffeomorphism  $\phi : Z \rightarrow Z'$ , then  $NH_T^{*,\diamond}(Z) \cong NH_T^{*,\diamond}(Z')$  as graded rings. Together with the proof given in the Appendix that the inertial cohomology of  $Z$  is isomorphic to the Chen–Ruan cohomology of the quotient orbifold, this justifies the claim in Sec. 2, i.e.  $H_{CR}^*(M)$  is indeed independent of the choice of regular value  $(\alpha, \alpha_{\mathbb{C}}) \in \mathfrak{t}^* \oplus \mathfrak{t}_{\mathbb{C}}^*$ . In particular, we may restrict without loss of generality, to the case  $(\alpha, 0)$ .

### 3.2. The product on $NH_T^{*,\diamond}(N)$ when $N$ is robustly equivariantly injective

In this section, we give a different description of the  $\smile$  product which will be easier to use for our computations. The  $T$ -space  $N$  is **robustly equivariantly injective** if the natural inclusion  $i : N^T \hookrightarrow N^t$  induces an injection in equivariant cohomology

$$i_t^* : H_T^*(N^t) \rightarrow H_T^*(N^T),$$

for all  $t \in T$ . When  $N$  satisfies this property, the product structure on  $NH_T^{*,\diamond}(N)$  can be described in terms of fixed point data and the local structure of the  $T$ -action near fixed points; see [5]. Robust equivariant injectivity is a strong condition: for instance, if  $T$  acts locally freely on  $N$ , then  $N^T = \emptyset$  and  $N$  certainly cannot be

robustly equivariantly injective. On the other hand, Hamiltonian  $T$ -spaces are an important source of examples of robustly equivariantly injective  $T$ -spaces.

For any component  $F$  of the fixed point set  $N^T$ ,  $T$  acts on the normal bundle to  $F$ . This representation splits into isotypic component under the action:

$$\nu(F, N) = \bigoplus_{\lambda \in \hat{T}} I_\lambda.$$

Let  $a \in NH_T^{*,t_1}(N)$  and  $b \in NH_T^{*,t_2}(N)$  be homogeneous classes in  $\diamond$ . Then  $i_{t_1}^*(a)$  and  $i_{t_2}^*(b)$  are classes in  $H_T^*((N^{t_1})^T)$  and  $H_T^*((N^{t_2})^T)$ , respectively. Both of these rings are identified naturally with  $H_T^*(N^T)$ . We define

$$i_{t_1}^*(a) \star i_{t_2}^*(b)|_F := i_{t_1}^*(a)|_F \cdot i_{t_2}^*(b)|_F \cdot \prod_{I_\lambda \subset \nu(F, N)} e(I_\lambda)^{a_\lambda(t_1) + a_\lambda(t_2) - a_\lambda(t_1 t_2)},$$

where  $e(I_\lambda) \in H_T^*(F)$  is the equivariant Euler class of  $I_\lambda$ , and all the products on the right-hand side are computed using the usual product in  $H_T^*(F)$ . Note that the exponent is either 0 (if  $a_\lambda(t_1) + a_\lambda(t_2) < 1$ ) or 1 (otherwise). By taking a sum over the connected components and by extending linearly, this defines a new product, which we call the  $\star$  **product**, on the image of  $i^*$  in  $\bigoplus_{t \in T} H^*(N^T)$ . When  $N$  is robustly equivariantly injective, the map  $i^*$  is injective, so the  $\star$  product uniquely defines a product on  $NH_T^{*,\diamond}(N)$ . By abuse of notation, we denote this product also as  $a \star b$ , for  $a, b \in NH_T^{*,\diamond}(N)$ .

The crucial fact, proven in [5], is that *these two product structures agree*, i.e.

$$a \smile b = a \star b,$$

when  $N$  is robustly equivariantly injective. Hence in the robustly equivariantly injective case we may, for the purposes of computation, work exclusively with the  $\star$  product. Note that  $N = T^*\mathbb{C}^n$  equipped with the  $T$ -action described in Sec. 2 is a Hamiltonian  $T$ -space, and in particular it is robustly equivariantly injective. We use this in Secs. 5 and 6 to simplify the combinatorics.

### 4. Surjection in Inertial Cohomology

Let  $M$  be a hypertoric variety as constructed in Sec. 2. Let  $Z := \mu_{HK}^{-1}(\alpha, \alpha_{\mathbb{C}}) \subseteq T^*\mathbb{C}^n$  be the level set of the hyperkähler moment map such that  $M = Z/T$ . In this section, we show that the map

$$\kappa_{NH} : NH_T^{*,\diamond}(T^*\mathbb{C}^n) \rightarrow NH_T^{*,\diamond}(Z) \tag{4.1}$$

induced by the inclusion  $i : Z \hookrightarrow T^*\mathbb{C}^n$  is a surjective ring homomorphism. In the Appendix we prove that the latter ring is isomorphic to the orbifold cohomology of  $M$  as (graded) rings, thus completing the proof of Theorem 1.1. An explicit description of both the domain and the kernel of  $\kappa_{NH}$ , provided in Sec. 5, will yield a combinatorial description of  $H_{CR}^*(M)$ . For the rest of the section, we will be largely following the outline of the proof of the symplectic case in [5]. However, there

are several new considerations in the hyperkähler case, which we will discuss as they arise.

We must first justify why the inertial cohomology of the level  $Z$  is defined. For this, it suffices to observe that the normal bundle to  $Z$  is trivial since  $Z$  is the preimage of a regular value of  $\mu_{HK}$ . Therefore, the complex bundle  $T(T^*\mathbb{C}^n)$  is a stabilization of  $TZ$ , so  $Z$  is also a stably complex  $T$ -space and  $NH_T^{*,\diamond}(Z)$  is well-defined.

Now consider the individual maps on Borel-equivariant cohomology

$$\kappa_{NH}^t : H_T^*((T^*\mathbb{C}^n)^t) \rightarrow H_T^*(Z^t) \tag{4.2}$$

induced by the inclusions  $Z^t \hookrightarrow (T^*\mathbb{C}^n)^t$ . Then we define the map on inertial cohomology to be the direct sum of the  $\kappa_t$ , i.e.

$$\kappa_{NH} := \bigoplus_{t \in T} \kappa_{NH}^t : NH_T^{*,\diamond}(T^*\mathbb{C}^n) \rightarrow NH_T^{*,\diamond}(Z). \tag{4.3}$$

This is *a priori* only a map of  $H_T^*(p)$ -modules, *not necessarily a ring homomorphism*. Indeed, given an inclusion of the  $T$ -fixed point set  $N^T \hookrightarrow N$  of a Hamiltonian  $T$ -space, the induced map  $NH_T^{*,\diamond}(N) \rightarrow NH_T^{*,\diamond}(N^T)$  on inertial cohomology does not, in general, preserve the product structure since the obstruction bundles are all trivial for  $NH_T^{*,\diamond}(N^T)$ . However, if a  $T$ -equivariant inclusion  $\iota : P \hookrightarrow N$  behaves well with respect to the fixed point sets  $N^t$  for all  $t \in T$ , then the obstruction bundles from Definition 3.3 also behave well, and the induced map on  $NH_T^{*,\diamond}$  is in fact a ring homomorphism. We quote the following [5, Proposition 5.1].

**Proposition 4.1 (Goldin–Holm–Knutson).** *Let  $N$  be a stably complex  $T$ -space. Let  $\iota : P \hookrightarrow N$  be a  $T$ -invariant inclusion and suppose also that  $P$  is transverse to any  $N^t$ ,  $t \in T$ . Then the map induced by inclusion  $\iota^* : NH_T^{*,\diamond}(N) \rightarrow NH_T^{*,\diamond}(P)$  is a ring homomorphism.*

Thus, in order to check that the map  $\kappa_{NH}$  is a ring homomorphism, it suffices to check that the level set  $Z$  is transverse to any  $(T^*\mathbb{C}^n)^t$ . We have the following general computation.

**Lemma 4.1.** *Let  $T$  be a compact torus, and let  $W$  be a hyperhamiltonian  $T$ -space with moment map  $\mu_{HK} = (\mu_1, \mu_2, \mu_3) : W \rightarrow (\mathfrak{t}^*)^3$ . Assume  $(\alpha_1, \alpha_2, \alpha_3)$  is a regular value of  $\mu_{HK}$ , and let  $Z$  denote the level set  $\mu_{HK}^{-1}(\alpha_1, \alpha_2, \alpha_3) \subseteq W$ . Then  $Z$  is transverse to  $W^t$  for any  $t \in T$ .*

**Proof.** The statement holds trivially if  $Z \cap W^t = \emptyset$ . We assume that  $Z \cap W^t \neq \emptyset$  and that  $W^t$  is connected; otherwise, we do the argument component by component. Let  $y \in Z \cap W^t$ , and let  $\iota_t : W^t \hookrightarrow W$  denote the inclusion. Since  $W^t$  is a fixed point set of a Hamiltonian  $T$ -action with respect to each symplectic form  $\omega_i$ ,  $W^t$  is itself a hyperhamiltonian  $T$ -submanifold of  $W$ , with moment map  $\iota_t^* \mu_{HK}$ .

Since  $(\alpha_1, \alpha_2, \alpha_3)$  is regular, the Lie algebra  $\text{Lie}(\text{Stab}(y)) \subseteq \mathfrak{t}$  of the stabilizer of  $y$  is 0. In order to prove the transversality, it suffices to prove that  $d(\iota_t^* \mu_{HK})_y$

is surjective. Since  $W^t$  is Kähler, and  $\text{Lie}(\text{Stab}(y)) = \{0\}$ ,  $d(\iota_t^* \mu_i)|_{J_i(T_y(T \cdot y))}$  is surjective onto  $\mathfrak{t}^*$  for each  $i$ , where  $T \cdot y$  is the  $T$ -orbit through  $y$  and  $T_y(T \cdot y)$  its tangent space at  $y$ . Moreover, since  $W^t$  is hyperkähler, the three subspaces  $J_i(T_y(T \cdot y))$  for  $i = 1, 2, 3$  are mutually orthogonal. In order to show that  $d(\iota_t^* \mu_{HK})_y$  is surjective onto  $(\mathfrak{t}^*)^3$ , it suffices to show that  $d(\iota_t^* \mu_i)(J_j(T_y(T \cdot y))) = 0$  for  $i \neq j$ . Without loss of generality, we take  $i = 2, j = 1$ . For any  $X, Y \in \mathfrak{t}$ ,

$$\begin{aligned} \langle d(\iota_t^* \mu_2)(J_1 X_y^\sharp), Y \rangle &= \omega_2(Y_y^\sharp, J_1 X_y^\sharp) && \text{by definition of a moment map} \\ &= -g(Y_y^\sharp, J_2 J_1 X_y^\sharp) && \text{compatibility between } g, \omega_2 \\ &= g(Y_y^\sharp, J_3 X_y^\sharp) && \text{quaternionic relation between the } J_i \\ &= -\omega_3(Y_y^\sharp, X_y^\sharp) && \text{compatibility between } g, \omega_3 \\ &= 0 && \text{since } T_y(T \cdot y) \text{ is isotropic with respect} \\ &&& \text{to } \omega_3. \end{aligned}$$

Thus  $d(\iota_t^* \mu_{HK})_y$  maps the span of the three subspaces  $J_i(T_y(T \cdot y)) \subseteq T_y W^t$  surjectively onto  $T_{(\alpha_1, \alpha_2, \alpha_3)}(\mathfrak{t}^*)^3 \cong (\mathfrak{t}^*)^3$ . This implies the level set  $Z = \mu_{HK}^{-1}(\alpha_1, \alpha_2, \alpha_3)$  is transverse to  $W^t$ . □

Proposition 4.1 together with Lemma 4.1 proves the following general fact.

**Proposition 4.2.** *Let  $T$  be a compact torus, and let  $W$  be a hyperhamiltonian  $T$ -space with moment map  $\mu_{HK} = (\mu_1, \mu_2, \mu_3) : W \rightarrow (\mathfrak{t}^*)^3$ . Assume  $(\alpha_1, \alpha_2, \alpha_3)$  is a regular value of  $\mu_{HK}$ , and let  $Z$  denote the level set  $\mu_{HK}^{-1}(\alpha_1, \alpha_2, \alpha_3) \subseteq W$ . Then the map on inertial cohomology induced by the inclusion  $Z \hookrightarrow W$ ,*

$$NH_T^{*,\diamond}(W) \rightarrow NH_T^{*,\diamond}(Z),$$

*is a ring homomorphism.*

In particular, in our case of hypertoric varieties, the map  $\kappa_{NH}$  defined in (4.3) is a ring homomorphism. Now it remains to show that  $\kappa_{NH} = \bigoplus_{t \in T} \kappa_{NH}^t$  is surjective. To do this, we show that

$$\kappa_{NH}^t : H_T^*((T^* \mathbb{C}^n)^t) \twoheadrightarrow H_T^*(Z^t) \tag{4.4}$$

is surjective for each  $t \in T$ . We begin with an analysis of these  $t$ -fixed point sets  $(T^* \mathbb{C}^n)^t$ . A direct calculation shows that

$$\begin{aligned} (T^* \mathbb{C}^n)^t &:= \{(z, w) \in T^* \mathbb{C}^n \mid z_i = w_i = 0 \text{ if } (\exp \lambda_i)(t) \neq 1\} \\ &\cong T^* \mathbb{C}^{S(t)} \end{aligned}$$

is a quaternionic affine subspace of  $T^* \mathbb{C}^n$ , where  $S(t) := \{i \in \{1, 2, \dots, n\} : (\exp \lambda_i)(t) = 1\} \subseteq \{1, 2, \dots, n\}$  and

$$\mathbb{C}^{S(t)} := \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : z_i = 0 \text{ if } i \notin S(t)\}. \tag{4.5}$$

In addition,  $T^* \mathbb{C}^{S(t)}$  is also a hyperhamiltonian  $T$ -space with moment map given by  $\iota_t^* \mu_{HK}$ , where  $\iota_t : (T^* \mathbb{C}^n)^t \hookrightarrow T^* \mathbb{C}^n$  denotes the inclusion. This computation allows us to conclude that (4.4) is the *ordinary* Kirwan map for the hypertoric subvariety

$T^*\mathbb{C}^{S(t)} \mathbin{////}_{(\alpha, \alpha_C)} T = Z^t/T$  of  $M$ . Since these maps are known to be surjective [11, 7], we have just proven the following.

**Theorem 4.1.** *Let  $T^*\mathbb{C}^n$  be a hyperhamiltonian  $T$ -space given by restriction of the standard hyperhamiltonian  $T^n$ -action on  $T^*\mathbb{C}^n$ , where the inclusion  $T \hookrightarrow T^n$  is determined as in Sec. 2. Then the map on inertial cohomology induced by the inclusion  $Z = \mu_{HK}^{-1}(\alpha, \alpha_C) \hookrightarrow T^*\mathbb{C}^n$ ,*

$$\kappa_{NH} : NH_T^{*,\diamond}(T^*\mathbb{C}^n) \rightarrow NH_T^{*,\diamond}(Z),$$

*is a surjective ring homomorphism.*

Inertial cohomology is a direct sum over infinitely many elements  $t \in T$ , so Theorem 4.1 is not at all amenable to computation. However, Theorem 4.1 can be substantially simplified for computational purposes (in particular, it can be made finite). We first establish some terminology. Suppose a torus  $T$  acts on a space  $Y$ . Suppose  $y \in Y$  and the stabilizer group  $\text{Stab}(y) \subseteq T$  is finite. Then we call  $\text{Stab}(y)$  a **finite stabilizer group**. Similarly, given a finite stabilizer group  $\text{Stab}(y)$ , we call any element  $t \in \text{Stab}(y)$  a **finite stabilizer (element)**. We let  $\Gamma$  denote the subgroup in  $T$  generated by finite stabilizers. In the case of a linear  $T$ -action on  $T^*\mathbb{C}^n$ , this is a finite subgroup of  $T$ , since the  $T$ -action is determined by a finite set of weights.

By Remark 3.2,

$$NH_T^{*,\Gamma}(T^*\mathbb{C}^n) := \bigoplus_{t \in \Gamma} NH_T^{*,t}(T^*\mathbb{C}^n) \subset NH_T^{*,\diamond}(T^*\mathbb{C}^n), \tag{4.6}$$

is a subring of  $NH_T^{*,\diamond}(T^*\mathbb{C}^n)$ . We call this the  $\Gamma$ -**subring**. We now show that  $\kappa_{NH}$  is still surjective when restricted to the  $\Gamma$ -subring.

**Theorem 4.2.** *Let  $T^*\mathbb{C}^n$  be a hyperhamiltonian  $T$ -space given by restriction of the standard hyperhamiltonian  $T^n$ -action on  $T^*\mathbb{C}^n$ , where the inclusion  $T \hookrightarrow T^n$  is determined as in Sec. 2. Let  $\Gamma$  be the subgroup in  $T$  generated by finite stabilizers. Then the map on the  $\Gamma$ -subrings of inertial cohomology induced by the inclusion  $Z = \mu_{HK}^{-1}(\alpha, \alpha_C) \hookrightarrow T^*\mathbb{C}^n$ ,*

$$\kappa_{NH}^\Gamma := \kappa_{NH}|_{NH_T^{*,\Gamma}(T^*\mathbb{C}^n)} : NH_T^{*,\Gamma}(T^*\mathbb{C}^n) \rightarrow NH_T^{*,\Gamma}(Z) \cong NH_T^{*,\diamond}(Z),$$

*is a surjective ring homomorphism.*

**Proof.** Since the level set  $Z$  is the preimage of a regular value of  $\mu_{HK}$ ,  $T$  acts locally freely on  $Z$ . In particular,  $Z^t = \emptyset$  if  $t \notin \Gamma$ . Hence for  $t \notin \Gamma$ , the map  $\kappa_{NH}^t : H_T^*((T^*\mathbb{C}^n)^t) \rightarrow H_T^*(Z^t)$  is automatically 0. Hence  $\kappa_{NH}^t$  does not contribute to the image of  $\kappa_{NH}$ , and  $\text{image}(\kappa_{NH}) = \text{image}(\kappa_{NH}|_{NH_T^{*,\Gamma}(T^*\mathbb{C}^n)})$ . In particular,  $\kappa_{NH}|_{NH_T^{*,\Gamma}(T^*\mathbb{C}^n)}$  is still surjective. □

By Theorem 4.2, we may restrict our attention to the  $\Gamma$ -subring and the restricted ring map  $\kappa_{NH}^\Gamma$ . The only remaining step to complete the proof of Theorem 1.1 is the isomorphism of the inertial cohomology  $NH_T^{*,\diamond}(Z) \cong NH_T^{*,\Gamma}(Z)$  of

the level set with the Chen–Ruan cohomology  $H_{CR}^*(M)$  of the quotient  $M = Z/T$ . This would then imply that, in order to compute  $H_{CR}^*(M)$ , it would suffice to compute the domain and kernel of  $\kappa_{NH}^\Gamma$ . This explicit computation is done in Sec. 5. The isomorphism  $NH_T^{*\diamond}(Z) \cong H_{CR}^*(M)$  mentioned above, for a general stably complex  $T$ -space with locally free  $T$ -action, is discussed in the compact case in [5, Sec. 4]; we place a detailed proof and its connection to the obstruction bundle in the algebraic geometry literature in the Appendix.

### 5. The Combinatorial Description of $H_{CR}^*(M)$

We now come to the main result of this manuscript. Using the inertial cohomology surjectivity result of Sec. 4 and the identification of inertial cohomology with Chen–Ruan orbifold cohomology in the Appendix, we give in this section an explicit description of the Chen–Ruan orbifold cohomology of hypertoric varieties in terms of the combinatorial data of the hyperplane arrangement  $\mathcal{H}$ . Here  $\mathcal{H} = \{H_i\}_{i=1}^n$  is a choice of simple affinization of the central arrangement  $\mathcal{H}_{\text{cent}}$  determining the hyperhamiltonian  $T$ -action on  $T^*\mathbb{C}^n$ , as detailed in Sec. 2. We then work out several concrete examples in Sec. 6.

We begin by stating our main theorem; for this, we must first set some notation. Let  $\mathcal{H}$  be a simple rational cooriented weighted hyperplane arrangement. Suppose that  $S \subseteq \{1, 2, \dots, n\}$  such that  $\{a_j\}_{j \in S^c}$  is linearly independent in  $\mathfrak{t}^d$ , where the  $a_j$  are integer normal vectors to the hyperplanes  $H_i$ . Then

$$\Gamma_S := \bigcap_{i \in S} \ker(\exp \lambda_i) \subseteq T \tag{5.1}$$

is a finite group. Let  $\Gamma$  be the finite subgroup in  $T$  generated by all such  $\Gamma_S$ . For an element  $t \in \Gamma$ , we define

$$S(t) := \{i : (\exp \lambda_i)(t) = 1\} \subseteq \{1, 2, \dots, n\}. \tag{5.2}$$

Let  $t_1, t_2 \in \Gamma$ . We define the following subsets of  $\{1, 2, \dots, n\}$ :

$$\begin{aligned} A(t_1, t_2) &:= \{i \in S(t_1)^c \cap S(t_2)^c \mid a_{\lambda_i}(t_1 t_2) = 0\}, \\ B(t_1, t_2) &:= \left\{ j \in S(t_1)^c \cap S(t_2)^c \mid \begin{array}{l} a_{\lambda_i}(t_1 t_2) \neq 0, \\ a_{\lambda_i}(t_1) + a_{\lambda_i}(t_2) - a_{\lambda_i}(t_1 t_2) = 0 \end{array} \right\}, \\ C(t_1, t_2) &:= \left\{ k \in S(t_1)^c \cap S(t_2)^c \mid \begin{array}{l} a_{\lambda_i}(t_1 t_2) \neq 0, \\ a_{\lambda_i}(t_1) + a_{\lambda_i}(t_2) - a_{\lambda_i}(t_1 t_2) = 1 \end{array} \right\}, \end{aligned} \tag{5.3}$$

where  $a_\lambda(t)$  is the logweight defined in Definition 3.2. Note that these sets partition the set of indices corresponding to lines with nontrivial action by  $t_1$  and  $t_2$ . With this notation in place, we may state our main theorem.

**Theorem 5.1.** *Let  $T^*\mathbb{C}^n$  be a hyperhamiltonian  $T$ -space given by restriction of the standard hyperhamiltonian  $T^n$ -action on  $T^*\mathbb{C}^n$ , where the inclusion  $T \hookrightarrow T^n$  is determined by the combinatorial data of  $\mathcal{H}_{\text{cent}}$  as in Sec. 2. Let  $\mathcal{H}$  be a simple*

affinization of  $\mathcal{H}_{\text{cent}}$ . Then for any regular value  $(\alpha, \alpha_{\mathbb{C}})$  the Chen–Ruan orbifold cohomology of the hyperkähler quotient  $M := T^*\mathbb{C}^n \mathbin{///}_{(\alpha, \alpha_{\mathbb{C}})} T$  is given by

$$H_{\text{CR}}^*(M) \cong \mathbb{Q}[u_1, \dots, u_n][\{\gamma_t\}_{t \in \Gamma}] / \mathcal{I} + \mathcal{J} + \mathcal{K} + \langle \gamma_{\text{id}} - 1 \rangle,$$

where the ideals  $\mathcal{I}, \mathcal{J}, \mathcal{K}$  are defined as follows. First,

$$\mathcal{I} = \left\langle \gamma_{t_1} \gamma_{t_2} - (-1)^{\sigma_{t_1 t_2}} \left( \prod_{i \in A(t_1, t_2)} u_i^2 \right) \left( \prod_{\substack{j \in \\ B(t_1, t_2) \sqcup C(t_1, t_2)}} u_j \right) \gamma_{t_1 t_2} \mid t_1, t_2 \in \Gamma \right\rangle$$

where  $\sigma_{t_1 t_2} = |A(t_1, t_2)| + |B(t_1, t_2)|$ . Second,

$$\mathcal{J} = \langle \text{im}(\beta^*) \rangle.$$

Finally,

$$\mathcal{K} = \sum_{t \in \Gamma} \left\langle \gamma_t \cdot \prod_{i \in L_t} u_i \mid \bigcap_{i \in L_t} H_i \cap \bigcap_{j \in S(t)^c} H_j = \emptyset \right\rangle,$$

where  $L_t$  denotes a (possibly empty) subset of  $S(t)$ .

The proof of Theorem 5.1 involves three steps. First, we must show that the subgroup generated by finite stabilizers defined in Sec. 4 is indeed the group  $\Gamma$  generated by the  $\Gamma_S$  in (5.1) above. Second, we prove that the ideals  $\mathcal{I}, \mathcal{J}$  above are exactly the relations which yield the inertial cohomology  $NH_T^{*, \Gamma}(T^*\mathbb{C}^n)$ . Finally, we show that the ideal  $\mathcal{K}$  exactly corresponds to the kernel of the inertial Kirwan map  $\ker(\kappa_{NH}^\Gamma)$ .

We begin with the first step, i.e. a description of the finite stabilizer group  $\Gamma$  associated to the given  $T$ -action on  $T^*\mathbb{C}^n$ . As a bonus, we also give an (easy to compute) description of the global orbifold structure groups that arise in the quotient hypertoric variety. Let  $M = T^*\mathbb{C}^n \mathbin{///}_{(\alpha, \alpha_{\mathbb{C}})} T$ , where the  $T$ -action on  $T^*\mathbb{C}^n$  is determined by  $\mathcal{H}_{\text{cent}}$ . We have the following.

**Proposition 5.1.** *Let  $T^*\mathbb{C}^n$  be a hyperhamiltonian  $T$ -space given by restriction of the standard hyperhamiltonian  $T^n$ -action on  $T^*\mathbb{C}^n$ , where the inclusion  $T \hookrightarrow T^n$  is determined by the combinatorial data of  $\mathcal{H}_{\text{cent}}$  as in Sec. 2. Let  $\{a_i\}_{i=1}^n$  be the positive normal vectors defining the hyperplanes in  $\mathcal{H}_{\text{cent}}$  and  $\lambda_i := \iota^* a_i$  as in (2.3).*

(1) *A subgroup of  $T$  is a finite stabilizer subgroup of a subvariety of  $T^*\mathbb{C}^n$  if and only if it is of the form*

$$\Gamma_S := \bigcap_{i \in S} \ker(\exp \lambda_i) \subseteq T, \tag{5.4}$$

where  $S \subseteq \{1, 2, \dots, n\}$  is such that  $\{a_j\}_{j \in S^c}$  is linearly independent in  $\mathfrak{t}^d$ . In particular, the subvariety  $M_S := T^*\mathbb{C}^S \mathbin{///}_{(\alpha, \alpha_{\mathbb{C}})} T$  has global orbifold structure group  $\Gamma_S$ .

(2) The subgroup  $\Gamma_S$  in (5.4) is isomorphic to

$$(\text{span}_{\mathbb{Q}}\{a_j\}_{j \in S^c} \cap \mathfrak{t}_{\mathbb{Z}}^d) / \text{span}_{\mathbb{Z}}\{a_j\}_{j \in S^c}.$$

(3) Any finite stabilizer  $t \in T$  occurs in a  $\Gamma_S$  for  $S$  such that  $\{a_j\}_{j \in S^c}$  forms a basis of  $\mathfrak{t}^d$ .

**Proof.** We begin with a general computation. Let  $(z, w) \in T^*\mathbb{C}^n$ . Recall that the action of the subtorus  $T \subseteq T^n$  is given by composing the homomorphism

$$T \rightarrow T^n, \quad t \mapsto ((\exp \lambda_1)(t), \dots, (\exp \lambda_n)(t)), \tag{5.5}$$

with the standard linear action of  $T^n$  on  $T^*\mathbb{C}^n$ . It is immediate that

$$\text{Stab}(z, w) = \{t \in T : \text{if either } z_i \neq 0 \text{ or } w_i \neq 0, \text{ then } (\exp \lambda_i)(t) = 1\}. \tag{5.6}$$

Now define  $S(z, w) := \{i | z_i \neq 0 \text{ or } w_i \neq 0\}$ . Then (5.6) becomes

$$\text{Stab}(z, w) = \bigcap_{i \in S(z, w)} \ker(\exp \lambda_i). \tag{5.7}$$

In particular,  $\text{Stab}(z, w)$  is finite if and only if the set  $\{\lambda_i\}_{i \in S(z, w)}$  spans  $\mathfrak{t}^*$ , or equivalently, the intersection  $\bigcap_{i \in S(z, w)} \ker(\lambda_i) = \{0\}$ . By the exactness of the sequence (2.1), this is equivalent to the condition that  $\{a_j\}_{j \in S(z, w)^c}$  is linearly independent in  $\mathfrak{t}^d$ . Conversely, given a subset  $S$  with  $\{a_j\}_{j \in S^c}$  linearly independent, any  $(z, w) \in T^*\mathbb{C}^n$  such that  $z_i = w_i = 0$  for  $i \notin S$ , and  $z_i \neq 0$  or  $w_i \neq 0$  for  $i \in S$ , will have stabilizer exactly  $\bigcap_{i \in S} \ker(\exp \lambda_i)$ . Moreover, the argument above immediately implies that  $M_S$  has global orbifold structure group  $\Gamma_S$ . This proves the first claim.

To prove the second claim, we will produce a map  $\varphi$  from  $(\text{span}_{\mathbb{Q}}\{a_j\}_{j \in S^c} \cap \mathfrak{t}_{\mathbb{Z}}^d)$  to  $T$ , which we will show takes values in  $\Gamma_S$ . For the remainder of this computation, we identify  $T$  with  $\ker(\beta)/(\ker(\beta) \cap \mathfrak{t}_{\mathbb{Z}}^n)$  by (2.1) and  $S^1$  with  $\mathbb{R}/\mathbb{Z}$ . In this language,  $[X] \in T$  is in  $\Gamma_S$  exactly when any representative  $X = \sum_i x_i \varepsilon_i \in \ker(\beta)$  of  $[X]$  has the property that  $x_i \in \mathbb{Z}$  for all  $i \in S$ . We begin by constructing the map  $\varphi$ . Let  $y \in \text{span}_{\mathbb{Q}}\{a_j\}_{j \in S^c} \cap \mathfrak{t}_{\mathbb{Z}}^d$ . Since  $\text{span}_{\mathbb{Z}}\{a_i\}_{i=1}^n = \mathfrak{t}_{\mathbb{Z}}^d$  by assumption, there exist linear combinations

$$y = \sum_{i=1}^n c_i a_i, \quad \text{and} \quad y = \sum_{j \in S^c} d_j a_j, \tag{5.8}$$

where  $c_i \in \mathbb{Z}, d_j \in \mathbb{Q}$ , and the second linear combination is unique. Let  $X = \sum_i x_i \varepsilon_i \in \mathfrak{t}^n$  where

$$x_k := \begin{cases} c_k & \text{if } k \in S \\ c_k - d_k & \text{if } k \in S^c. \end{cases}$$

Then by construction  $x$  represents an element in  $\Gamma_S$ , and we define  $\varphi(y) := [x] \in T$ . A different choice of  $\mathbb{Z}$ -linear combination in (5.8) yields the same  $[x]$ , so  $\varphi$  is well-defined. Furthermore, by definition, if  $y \in \text{span}_{\mathbb{Z}}\{a_j\}_{j \in S^c}$ ,  $\varphi(y)$  is trivial in



$T$ , so  $\varphi$  also factors through the quotient

$$(\text{span}_{\mathbb{Q}}\{a_j\}_{j \in S^c} \cap \mathfrak{t}_{\mathbb{Z}}^d) / \text{span}_{\mathbb{Z}}\{a_j\}_{j \in S^c}.$$

The map  $\varphi$  preserves additive structures, hence is a homomorphism. Furthermore,  $\varphi$  is an injection since if  $\varphi(y) \in \ker(\beta) \cap \mathfrak{t}_{\mathbb{Z}}^n$ , then the coefficients  $d_j$  in (5.8) are integers, and hence  $y \in \text{span}_{\mathbb{Z}}\{a_j\}_{j \in S^c}$ . Finally, to see that  $\varphi$  is surjective, let  $X \in \ker(\beta)$  be a representative for an element in  $\Gamma_S$ , with coordinates  $c_i$  for  $i \in S$ ,  $x_j$  for  $j \in S^c$ . Then  $y := \sum_{i \in S} c_i a_i$  has the property that  $\varphi(y) = [X]$ , so  $\varphi$  is surjective. Hence  $\varphi$  is an isomorphism, as desired.

Finally, since we always have

$$\bigcap_{i \in S'} \ker(\exp \lambda_i) \subseteq \bigcap_{i \in S} \ker(\exp \lambda_i)$$

for any  $S \subseteq S'$ , in order to identify the finite stabilizer elements in  $T$ , it suffices to consider the *minimal* subsets  $S$  such that  $\bigcap_{i \in S} \ker \lambda_i = \{0\}$ , or equivalently, *maximal* linearly independent sets  $\{a_j\}_{j \in S^c}$ , i.e. bases of  $\mathfrak{t}^d$ . This proves the final claim.  $\square$

Thus, in order to compute  $\Gamma$ , it suffices to find the subsets  $\{a_j\}_{j \in S^c}$  in  $\{a_i\}_{i=1}^n$  which form a basis of  $\mathfrak{t}^d$ . We also note that the subvarieties  $M_S$  map under the moment map for  $M$  to the intersection of the hyperplanes  $\bigcap_{j \in S^c} H_j$ , so can easily be identified in the combinatorial picture using  $\mathcal{H}$ .

We now proceed to the second step, i.e. we describe the product structure on  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$ .

**Proposition 5.2.** *Let  $T^*\mathbb{C}^n$  be a hyperhamiltonian  $T$ -space given by restriction of the standard hyperhamiltonian  $T^n$ -action on  $T^*\mathbb{C}^n$ . Let  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$  be the  $\Gamma$ -subring of the inertial cohomology ring  $NH_T^{*,\diamond}(T^*\mathbb{C}^n)$ , and let  $\lambda_i := \iota^*u_i$  as in (2.3). Then, as a graded  $H_T^*(pt; \mathbb{Q})$ -algebra,*

$$NH_T^{*,\Gamma}(T^*\mathbb{C}^n) \cong \mathbb{Q}[u_1, u_2, \dots, u_n][\{\gamma_t\}_{t \in \Gamma}] / \mathcal{I} + \mathcal{J} + \langle \gamma_{\text{id}} - 1 \rangle,$$

where the ideal  $\mathcal{I}$  is generated by the relations

$$\gamma_{t_1} \gamma_{t_2} = (-1)^{|A(t_1, t_2)| + |B(t_1, t_2)|} \left( \prod_{i \in A(t_1, t_2)} u_i^2 \right) \left( \prod_{j \in B(t_1, t_2) \sqcup C(t_1, t_2)} u_j \right) \gamma_{t_1 t_2}, \quad (5.9)$$

with the sets  $A(t_1, t_2), B(t_1, t_2), C(t_1, t_2)$  as defined in (5.3), and  $\mathcal{J} = \langle \text{im}(\beta^*) \rangle$ .

**Remark 5.1.** The grading is given by  $\deg u_i = 2$  for all  $i$ , and  $\deg \gamma_t = 2 \text{age}(t)$ , as specified in the Appendix.

**Proof.** Recall that the  $\Gamma$ -subring  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$  is by definition given by

$$NH_T^{*,\Gamma}(T^*\mathbb{C}^n) := \bigoplus_{t \in \Gamma} H_T^*((T^*\mathbb{C}^n)^t).$$

Since each  $t$ th graded piece is the  $T$ -equivariant cohomology of a contractible space, it has a single generator as a  $H_T^*(pt)$ -module. Let  $\gamma_t$  denote the element

in  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$  which is equal to 0 for each  $h$ -graded piece with  $h \neq t, h \in \Gamma$ , and which is equal to the generator  $1 \in H_T^*((T^*\mathbb{C}^n)^t) \cong H_T^*(pt)$  in the  $t$ th graded piece. Then  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$  is generated as a  $H_T^*(pt)$ -module by these  $\{\gamma_t\}_{t \in \Gamma}$ . Hence in order to determine the multiplicative structure, it suffices to find the product relations among these generators  $\gamma_t, t \in \Gamma$ . Also, by the exact sequence (2.1), we may identify

$$H_T^*(pt; \mathbb{Q}) \cong H_{T^n}^*(pt; \mathbb{Q})/\mathcal{J} \cong \mathbb{Q}[u_1, \dots, u_n]/\mathcal{J}.$$

Since  $T^*\mathbb{C}^n$  is robustly equivariantly injective, we may compute all products in terms of the  $\star$  product instead of the  $\smile$  product, as was explained in Sec. 3.2. By our assumptions on  $\mathcal{H}$ , all the weights  $\lambda_i$  defining the action of  $T$  on  $T^*\mathbb{C}^n$  are non-zero, and hence the only  $T$ -fixed point is the origin  $\{0\} \in T^*\mathbb{C}^n$ . The  $T$ -weights of the action on the normal bundle  $\nu(F, N)$  to  $F := (T^*\mathbb{C}^n)^T = \{0\}$  are the  $2n$  weights  $\{\pm \lambda_i\}_{i=1}^n$ . Since the weights come in pairs, the definition of the  $\star$  product yields

$$\gamma_{t_1} \star \gamma_{t_2} = \gamma_{t_1 t_2} \cdot \prod_{i=1}^n ((\lambda_i)^{a_{\lambda_i}(t_1) + a_{\lambda_i}(t_2) - a_{\lambda_i}(t_1 t_2)} \cdot (-\lambda_i)^{a_{-\lambda_i}(t_1) + a_{-\lambda_i}(t_2) - a_{-\lambda_i}(t_1 t_2)}).$$

If  $i \in S(t_1) \cup S(t_2)$ , then either  $a_{\lambda_i}(t_1) = 0$  or  $a_{\lambda_i}(t_2) = 0$ , and the corresponding exponent is 0. Suppose  $i \in S(t_1)^c \cap S(t_2)^c$ . We now take cases. Suppose that  $i \in A(t_1, t_2)$ . In this case,  $a_{-\lambda_i}(t_\ell) = 1 - a_{\lambda_i}(t_\ell)$  for either  $\ell = 1$  or  $2$ ,  $a_{\lambda_i}(t_1) + a_{\lambda_i}(t_2) = 1$  and the  $i$ th term in the product above is  $-\lambda_i^2$ . Similar computations show that if  $j \in B(t_1, t_2)$ , then the  $j$ th term is equal to  $-\lambda_j$ , and if  $k \in C(t_1, t_2)$ , then the  $k$ th term is  $\lambda_k$ . Finally, given the identification of  $H_T^*(pt; \mathbb{Q})$  with  $H_{T^n}^*(pt; \mathbb{Q})/\mathcal{J}$ , a representative of  $\lambda_i \in H_T^2(pt; \mathbb{Q})$  is given by  $u_i \in H_{T^n}^2(pt; \mathbb{Q})$ . □

The third and final step is to determine the kernel of the inertial Kirwan map  $\kappa_{NH}^\Gamma$ . Since  $\kappa_{NH}^\Gamma = \bigoplus_{t \in \Gamma} \kappa_{NH}^{t}$  is a direct sum of maps  $\kappa_{NH}^{t} : NH_T^{*,t}((T^*\mathbb{C}^n)^t) \rightarrow NH_T^{*,t}(Z^t)$  for  $t \in \Gamma$ , it suffices to compute the kernel of each  $\kappa_{NH}^{t}$  separately.

Suppose  $t \in \Gamma$ . Then, as observed in Sec. 4,  $\kappa_{NH}^{t}$  is the Kirwan map in usual cohomology  $H^*(-; \mathbb{Q})$  for the hyperkähler Delzant construction of an orbifold hypertoric variety. This map is known to be surjective and the kernel has been explicitly computed. We quote the following.

**Theorem 5.2 (Hausel–Sturmfels).** *Let  $T^*\mathbb{C}^n$  be a hyperhamiltonian  $T$ -space given by restriction of the standard linear hyperhamiltonian  $T^n$ -action on  $T^*\mathbb{C}^n$ , where the inclusion  $T \hookrightarrow T^n$  is determined by the combinatorial data of  $\mathcal{H}_{\text{cent}}$  as in Sec. 2 and let  $\mathcal{H}$  be a simple affinization of  $\mathcal{H}_{\text{cent}}$ . Then the ordinary cohomology ring of the orbifold hypertoric variety  $M = T^*\mathbb{C}^n //_{(\alpha, \alpha_C)} T$  is given by*

$$H^*(M; \mathbb{Q}) \cong \mathbb{Q}[u_1, \dots, u_n] / \mathcal{J} + \left\langle \prod_{i \in L} u_i \mid \bigcap_{i \in L} H_i = \emptyset \right\rangle, \tag{5.10}$$

where  $L$  denotes a subset of  $\{1, 2, \dots, n\}$ .

We will apply Theorem 5.2 to the hyperhamiltonian action of  $T$  on  $T^*\mathbb{C}^{S(t)}$  for each  $t \in \Gamma$ . Since the expression in (5.10) uses the combinatorial data of the hyperplane arrangement corresponding to this action, our first task will be to describe explicitly the arrangement  $\mathcal{H}_t$  for each  $t \in \Gamma$  in terms of the original  $\mathcal{H}$ . Since the  $T$ -action is defined as a restriction of the original  $T$ -action on  $T^*\mathbb{C}^n$ , the action on  $T^*\mathbb{C}^{S(t)}$  is given by the composition  $\iota_t := \pi_t \circ \iota$ ,

$$\begin{array}{ccccc} & & \overset{\iota_t}{\curvearrowright} & & \\ \mathfrak{t} & \xrightarrow{\iota} & \mathfrak{t}^n & \xrightarrow{\pi_t} & \mathfrak{t}^{S(t)} \end{array}$$

where  $\iota$  is the inclusion coming from the original Delzant sequence (2.1), and  $\pi_t$  is the natural projection to the subspace, given by  $\varepsilon_i \mapsto [\varepsilon_i], \forall i \in S(t), \varepsilon_j \mapsto 0, \forall j \in S(t)^c$ . A simple linear algebra argument together with the commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{t} & \xrightarrow{\iota} & \mathfrak{t}^n & \xrightarrow{\beta} & \mathfrak{t}^{d=n-k} \\ & & \searrow \iota_t & & \downarrow \pi_t & & \downarrow \\ & & \mathfrak{t}^{S(t)} \cong \mathfrak{t}^n / \text{span} \langle \varepsilon_i \rangle_{i \in S(t)^c} & \xrightarrow{\beta_t} & \mathfrak{t}^d / \text{span} \langle a_i \rangle_{i \in S(t)^c} & & \end{array} \tag{5.11}$$

(where the top exact sequence is that in (2.1), the right vertical arrow is the natural projection, and  $\beta_t$  is the composition of  $\beta$  with the natural projection) shows that the map  $\iota_t$  fits into the exact sequence

$$0 \longrightarrow \mathfrak{t} \xrightarrow{\iota_t} \mathfrak{t}^{S(t)} \xrightarrow{\beta_t} \mathfrak{t}^d / \text{span} \langle a_i \rangle_{i \in S(t)^c} \longrightarrow 0. \tag{5.12}$$

From this sequence, we will be able to deduce the structure of the arrangement  $\mathcal{H}_t$ . We note, however, the sequence (5.12) is not necessarily a standard Delzant exact sequence as in Sec. 2. This is because it is possible to have  $\beta_t([\varepsilon_i]) = 0$  for some  $i \in S(t)$ , whereas this does not occur for a standard Delzant construction. This poses no serious problems, as will be discussed in more detail later. The relevant combinatorial data must therefore be contained in the *non-zero* images,  $[a_i] \neq 0, i \in S(t)$ .

Using (5.12), we may now explicitly describe the hyperplane arrangement  $\mathcal{H}_t$  for  $t \in \Gamma$ . First,  $\mathcal{H}_t$  sits naturally in the dual of the Lie algebra  $\mathfrak{t}^d / \text{span} \langle a_j \rangle_{j \in S(t)^c}$ , which is a subspace of  $(\mathfrak{t}^d)^*$ . Specifically, it is the annihilator of the  $|S(t)^c|$ -dimensional subspace  $\text{span} \langle a_j \rangle_{j \in S(t)^c}$ . Hence, up to an affine translation, it may be identified with the intersection

$$\bigcap_{j \in S(t)^c} H_j \subseteq (\mathfrak{t}^d)^*, \tag{5.13}$$

where  $H_j$  is the hyperplane orthogonal to  $a_j$  in the original hyperplane arrangement  $\mathcal{H}$ . Moreover, by analyzing the dimensions of  $T^d$ -orbits in the subvariety

$M_{S(t)} \subseteq M$ , it is straightforward to see that the affine hyperplanes in  $\mathcal{H}_t$  are exactly given by the intersections

$$H_i \cap \bigcap_{j \in S(t)^c} H_j \subseteq \bigcap_{j \in S(t)^c} H_j, \tag{5.14}$$

where  $i \in S(t)$  such that  $a_i \notin \text{span} \langle a_j \rangle_{j \in S(t)^c}$ .

The only indices  $i \in S(t)$  which give non-empty hyperplanes in  $\mathcal{H}_t$  are those for which  $[a_i] = \beta([\varepsilon_i]) \neq 0$ . Thus, in addition to the standard Delzant construction for  $\mathcal{H}_t$  with certain basis vectors  $[\varepsilon_i]$  mapping to the corresponding normal vector  $[a_i] \neq 0$  defining a hyperplane in  $\mathcal{H}_t$ , we have in this case also some extra basis elements corresponding to the  $i \in S(t)$  for which  $\beta[\varepsilon_i] = 0$ . This poses no problems, because any such extra indices correspond to a subtorus of  $T$  acting *standardly* on a quaternionic affine space, the moment map for which has level sets precisely equal to group orbits and hence has trivial hyperkähler quotient. In particular, the addition of such extra indices leaves the corresponding hypertoric variety topologically unchanged, allowing us (with only slight modifications) to use the known theorems for hypertoric varieties built via a standard Delzant construction.

We have the following proposition.

**Proposition 5.3.** *Let  $T^*\mathbb{C}^n$  be a hyperhamiltonian  $T$ -space given by restriction of the standard hyperhamiltonian  $T^n$ -action on  $T^*\mathbb{C}^n$ , where the inclusion  $T \hookrightarrow T^n$  is determined by the combinatorial data of  $\mathcal{H}_{\text{cent}}$  as in Sec. 2, and let  $\mathcal{H}$  be a simple affinization of  $\mathcal{H}_{\text{cent}}$ . Let  $t \in \Gamma$ . Then the ordinary cohomology of the hypertoric subvariety  $M_S = T^*\mathbb{C}^{S(t)} // T$  is given by*

$$H^*(M_S; \mathbb{Q}) \cong \mathbb{Q}[u_1, u_2, \dots, u_n] / \mathcal{J} + \mathcal{K}_t,$$

where

$$\mathcal{K}_t = \left\langle \prod_{i \in L_t} u_i \mid \bigcap_{i \in L_t} H_i \cap \bigcap_{j \in S(t)^c} H_j = \emptyset \right\rangle, \tag{5.15}$$

where  $L_t$  denotes a (possibly empty) subset of  $S(t)$ , and  $\mathcal{J} = \langle \text{im}(\beta)^* \rangle$ .

**Proof.** We begin by observing that the domain  $H_T^*(T^*\mathbb{C}^{S(t)}) \cong H_T^*(pt)$  of the map  $\kappa_{NH}^t$  may also be identified as  $H_{T^n}^*(pt) / \langle \text{im}(\beta^*) \rangle \cong \mathbb{Q}[u_1, u_2, \dots, u_n] / \langle \text{im}(\beta^*) \rangle$ , by (2.1).

A subtlety that arises here is the presence of global stabilizers for the  $T$ -action on the subsets  $(T^*\mathbb{C}^n)^t = T^*\mathbb{C}^{S(t)}$  for  $t \in \Gamma$  a non-trivial finite stabilizer. Clearly, if  $t \neq \text{id}$ , then by definition  $(T^*\mathbb{C}^n)^t$  has some non-trivial global stabilizer  $\Gamma_t$ . Hence the  $T$ -action on  $T^*\mathbb{C}^{S(t)}$  is not effective and in particular does not arise from a standard Delzant construction (since any such is effective). However, since  $\Gamma_t$  is finite,  $T/\Gamma_t$  is again a torus of dimension  $\dim(T)$ , and the inclusion maps on the level of Lie algebras are identical. The same holds at the level of cohomology rings, and hence the computation with global finite stabilizer is identical to the

computation in the usual hyperkähler Delzant construction. (Put another way, the essential data for the computation is in the maps on Lie algebras.)

Putting together the description given in (5.13) and (5.14) of the hyperplane arrangement  $\mathcal{H}_t$  associated to (5.12), Theorem 5.2, and the commutative diagram (5.11), we see that the kernel of  $\kappa_{NH}^t$  is generated by the relations given in (5.15). Note that if  $i \in S(t)$  such that  $\beta[\varepsilon_i] = 0$ , then

$$H_i \cap \bigcap_{j \in S(t)^c} H_j = \emptyset,$$

so  $u_i \in \mathcal{K}_t$ . Finally, we observe that if  $\bigcap_{j \in S(t)^c} H_j = \emptyset$  (for instance, if  $t \in \Gamma$  is not a finite stabilizer) then  $\kappa_{NH}^t \equiv 0$  since in this case  $Z^t = \emptyset$ . This in particular implies that we must have  $\gamma_t \in \mathcal{K}_t$ , which is implied by our convention that we can take  $L_t = \emptyset$  in the relations above and hence  $\gamma_t \cdot 1 = \gamma_t \in \mathcal{K}_t$ . The result follows.  $\square$

We may now prove our main theorem, which gives a full combinatorial description of the Chen–Ruan orbifold cohomology of the orbifold hypertoric variety  $M$ .

**Proof of Theorem 5.1.** We will give a description of the Chen–Ruan orbifold cohomology of  $M$  as a quotient of  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$  by the kernel of  $\kappa_{NH}^\Gamma$ . From Proposition 5.2, we have already seen that  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$  can be written as

$$NH_T^{*,\Gamma}(T^*\mathbb{C}^n) \cong \mathbb{Q}[u_1, u_2, \dots, u_n][\{\gamma_t\}_{t \in \Gamma}]/\mathcal{I} + \mathcal{J} + \langle \gamma_{\text{id}} - 1 \rangle.$$

Thus it remains to describe each piece of the kernel,  $\mathcal{K} = \ker(\kappa_{NH}^\Gamma) = \bigoplus_{t \in \Gamma} \ker(\kappa_{NH}^t)$ . Proposition 5.3 implies that  $\gamma_t \star \mathcal{K}_t \subseteq \mathcal{K}$  where  $\mathcal{K}_t$  is defined in (5.15) and is here considered as an ideal in  $NH_T^{*,\text{id}}(T^*\mathbb{C}^n)$ . Note that  $\gamma_t \star \mathcal{K}_t$  is a subset of the  $t$ th graded piece  $NH_T^{*,t}(T^*\mathbb{C}^n)$ .

This concludes the proof except for one subtlety: in Proposition 5.3, we described ideal generators for  $\mathcal{K}_t$  with respect to the *standard* ring structure of Borel-equivariant cohomology, whereas in Theorem 5.1, we present generators with respect to the  $\star$  (or equivalently  $\smile$ ) product on  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$ . Thus, given generators in the standard product, it is not immediate that their union (multiplied by appropriate  $\gamma_t$ ) would yield ideal generators for  $\mathcal{K}$  in the  $\star$  product. However, the id-graded piece  $NH_T^{*,\text{id}}(T^*\mathbb{C}^n) \cong H_T^*(pt)$  is a subring of  $NH_T^{*,\Gamma}(T^*\mathbb{C}^n)$  in the  $\star$  product, and multiplication in the  $\star$  product of elements in  $NH_T^{*,\text{id}}(T^*\mathbb{C}^n)$  and  $NH_T^{*,t}(T^*\mathbb{C}^n)$  agrees with the standard  $H_T^*(pt)$ -module structure on  $H_T^*((T^*\mathbb{C}^n)^t)$  in Borel-equivariant cohomology. Since Theorem 5.2 gives  $H_T^*(pt)$ -module generators for each  $\mathcal{K}_t$ , the result follows.  $\square$

### 6. Examples

We compute several explicit examples in this section to illustrate our methods. Throughout, we identify Lie algebras with their dual spaces using the standard inner product. When illustrating the hyperplane arrangements, we will shade the intersection of the positive half-spaces corresponding to the cooriented hyperplanes.

**6.1. A hyperkähler analogue of an orbifold  $\mathbb{P}^2$**

We begin with an example in which the corresponding Kähler toric variety is an orbifold  $\mathbb{P}^2$ . Let  $\mathcal{H}$  be the hyperplane arrangement depicted in Fig. 1 and denote the corresponding hypertoric variety by  $M$ . In this example,  $n = 3, d = 2, k = 1$ . Here we will take the normal vectors to the hyperplanes to be primitive.

With respect to the standard bases in  $\mathfrak{t}^3$  and  $\mathfrak{t}^2$ , the map  $\beta$  in (2.1) is given by

$$\beta = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix},$$

where the  $i$ th column is the vector  $a_i$  normal to the  $i$ th hyperplane in Fig. 1. By Proposition 5.1, the single orbifold point maps to the intersection of the hyperplanes  $H_2$  and  $H_3$ . The kernel of  $\beta$  is given by the span of the single vector  $(2, 1, 1)$  in  $\mathfrak{t}^3$ . Hence the  $S^1$ -action on  $T^*\mathbb{C}^3$  with respect to which we take a hyperkähler quotient is induced by the linear action of  $S^1$  on  $\mathbb{C}^3$  with weights  $2, 1, 1$  on the three coordinates, respectively. In particular, it is immediate that the finite stabilizer subgroup  $\Gamma$  is just  $\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ . We compute the following table of logweights; the quantity  $2\text{age}(t)$  is the degree of the corresponding generator as in (A.5).

$t$	$a_2(t)$	$a_1(t)$	$a_1(t)$	$2 \text{ age}(t)$	Generator of $NH_{S^1}^{*,t}(T^*\mathbb{C}^3)$
id	0	0	0	0	$\gamma_{\text{id}}$
-1	0	$\frac{1}{2}$	$\frac{1}{2}$	4	$\gamma_{-1}$

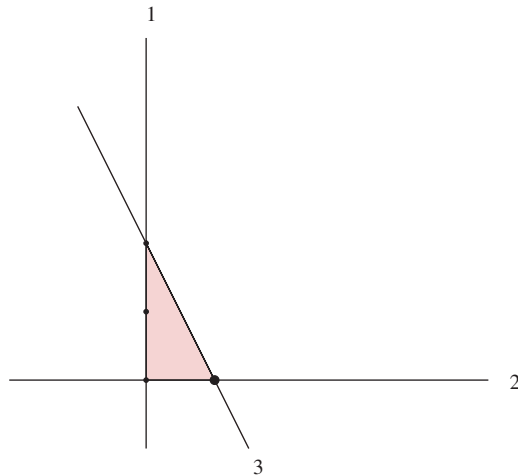
(6.1)


Fig. 1. An example of an orbifold hypertoric variety obtained by reducing  $\mathbb{H}^3$  by  $S^1$ . The corresponding Kähler toric variety is a  $\mathbb{P}^2$  with a single orbifold point, which maps to  $H_2 \cap H_3$ .

Since there is only one non-trivial generator  $\gamma_{-1}$  in  $NH_{S^1}^{*\Gamma}(T^*\mathbb{C}^3)$  as a  $H_{S^1}^*(pt)$ -module, we only need to compute a single relation of the form (5.9), namely, the product of  $\gamma_{-1}$  with itself. Since  $t^2 = \text{id} = 1$  for  $t = -1$ , we also have

$$A(-1, -1) = \{2, 3\}, \quad B(-1, -1) = \emptyset, \quad C(-1, -1) = \emptyset,$$

as can be computed from the definitions (5.3), and so we have

$$\gamma_{-1}^2 - u_2^2 u_3^2 \in \mathcal{I}.$$

The ideal  $\mathcal{J}$  of linear relations can be deduced from the matrix of  $\beta$  to be

$$\mathcal{J} = \langle u_1 - 2u_3, u_2 - u_3 \rangle.$$

Finally, the ideal  $\mathcal{K} = \ker(\kappa_{NH}^\Gamma)$  may be computed via the two pieces  $\ker(\kappa_{NH}^{\text{id}}), \ker(\kappa_{NH}^{-1})$ . We have from Proposition 5.3 that

$$\mathcal{K}_{\text{id}} = \langle u_1 u_2 u_3 \rangle, \quad \text{and} \quad \mathcal{K}_{-1} = \langle \gamma_{-1} u_1 \rangle,$$

from which we conclude that

$$H_{CR}^*(M) \cong \mathbb{Q}[u_1, u_2, u_3, \gamma_{\text{id}}, \gamma_{-1}] / \left\langle \begin{array}{l} \gamma_{-1}^2 - u_2^2 u_3^2, u_1 - 2u_3, u_2 - u_3, \\ u_1 u_2 u_3, \gamma_{-1} u_1, \gamma_{\text{id}} - 1 \end{array} \right\rangle,$$

which is easily shown to be isomorphic to

$$H_{CR}^*(M) \cong \mathbb{Q}[u, \gamma] / \langle u^3, \gamma^2, \gamma u \rangle,$$

where  $\text{deg}(u) = 2, \text{deg}(\gamma) = 4$ . From this it is straightforward to compute that the orbifold Poincaré polynomial for  $M$  is given by

$$P_{\text{orb}}(t, M) = 1 + t^2 + 2t^4,$$

so the orbifold Euler characteristic is 4.

**Remark 6.1.** Let  $M_n$  denote the hypertoric variety associated to the more general case in which  $a_3 = (-n, -1)$  (so the above case is  $n = 2$ ). The underlying Kähler toric variety is a weighted  $\mathbb{P}^2$  with a single orbifold point with orbifold structure group  $\mathbb{Z}/n\mathbb{Z}$ . An analogous computation yields the orbifold Poincaré polynomial

$$P_{\text{orb}}(t, M_n) = 1 + t^2 + nt^4,$$

so  $M_n$  has orbifold Euler characteristic  $n + 2$ .

### 6.2. A quotient of $T^*\mathbb{C}^4$ by a $T^2$

We continue with an example in which the corresponding Kähler toric variety is a smooth  $\mathbb{P}^2$ , but now we add an extra hyperplane which introduces an orbifold point in the hypertoric variety. Let  $\mathcal{H}$  be the hyperplane arrangement depicted in Fig. 2 and denote by  $M$  the corresponding hypertoric variety. In this example,  $n = 4, d = 2, k = 2$ . We take primitive normals to these hyperplanes.

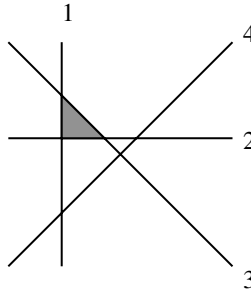


Fig. 2. An example of an orbifold hypertoric variety obtained by reducing  $\mathbb{H}^4$  by  $T^2$ . The intersection  $H_3 \cap H_4$  corresponds to the orbifold point.

The map  $\beta$  is given by the matrix

$$\beta = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

By Proposition 5.1, the single orbifold point maps to the intersection of the hyperplanes  $H_3$  and  $H_4$ . The kernel of  $\beta$  is given by the Lie subalgebra  $\mathfrak{t}^k = \mathfrak{t}^2 \subseteq \mathfrak{t}^4$  given by the span of the vectors  $(1, 1, 1, 0)$  and  $(1, -1, 0, 1)$  in  $\mathfrak{t}^4 \cong \mathbb{R}^4$ . Therefore, the  $T$ -action on  $T^*\mathbb{C}^4$  with respect to which we take a hyperkähler quotient is given by

$$(t_1, t_2) \cdot (z, w) = (t_1 t_2 z_1, t_1 t_2^{-1} z_2, t_1 z_3, t_2 z_4, t_1^{-1} t_2^{-1} w_1, t_1^{-1} t_2 w_2, t_1^{-1} w_3, t_2^{-1} w_4),$$

where  $z = (z_1, z_2, z_3, z_4), w = (w_1, w_2, w_3, w_4)$ , and here we have chosen an identification of the kernel of  $\exp(\beta)$  with the standard 2-torus  $T^2$ .

The finite stabilizer group  $\Gamma$  may now be computed as follows. The weights  $\{\lambda_i\}_{i=1}^4$  for the  $T^2$ -action are given by  $\lambda_1 = (1, 1), \lambda_2 = (1, -1), \lambda_3 = (1, 0), \lambda_4 = (0, 1)$ . The only minimal spanning subset which leads to a non-trivial stabilizer is  $\{\lambda_1, \lambda_2\}$ , and the stabilizer subgroup is generated by the element  $(-1, -1) \in T^2$ . Hence  $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ . Then immediately  $S(\text{id}) = \{1, 2, 3, 4\}$  and  $S((-1, -1)) = \{1, 2\}$ . We will also use the following table.

$t$	$a_{(1,1)}(t)$	$a_{(1,-1)}(t)$	$a_{(1,0)}(t)$	$a_{(0,1)}(t)$	$2 \text{ age}(t)$	Generator of $NH_T^{*,t}(T^*\mathbb{C}^4)$
id	0	0	0	0	0	$\gamma_{\text{id}}$
$(-1, -1)$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	4	$\gamma_{(-1,-1)}$

(6.2)

As in the previous example, we only need to compute a single relation of the form (5.9), namely, the product of  $\gamma_{(-1,-1)}$  with itself. We have

$$\begin{aligned} A((-1, -1), (-1, -1)) &= \{3, 4\}, \\ B((-1, -1), (-1, -1)) &= \emptyset, \\ C((-1, -1), (-1, -1)) &= \emptyset, \end{aligned}$$



and so

$$\gamma_{(-1,-1)}^2 - u_3^2 u_4^2 \in \mathcal{I}.$$

The ideal of linear relations is

$$\mathcal{J} = \langle u_1 - u_3 - u_4, u_2 - u_3 + u_4 \rangle.$$

Again as in the previous example, the ideal  $\mathcal{K} = \ker(\kappa_{NH}^\Gamma)$  may be computed via the two pieces  $\ker(\kappa_{NH}^{\text{id}}), \ker(\kappa_{NH}^{(-1,-1)})$ . We have

$$\begin{aligned} \mathcal{K}_{\text{id}} &= \langle u_2 u_3 u_4, u_1 u_3 u_4, u_1 u_2 u_4, u_1 u_2 u_3 \rangle, \quad \text{and} \\ \mathcal{K}_{(-1,-1)} &= \langle \gamma_{(-1,-1)} u_1, \gamma_{(-1,-1)} u_2 \rangle. \end{aligned}$$

We conclude

$$H_{CR}^*(M) \cong \mathbb{Q}[u_1, u_2, u_3, u_4, \gamma_{\text{id}}, \gamma_{(-1,-1)}] / I$$

where

$$I = \left\langle \begin{array}{l} \gamma_{(-1,-1)}^2 - u_3^2 u_4^2, u_1 - u_3 - u_4, u_2 - u_3 + u_4, \\ u_1 u_3 u_4, u_2 u_3 u_4, u_1 u_2 u_3, u_1 u_2 u_4 \\ \gamma_{(-1,-1)} u_1, \gamma_{(-1,-1)} u_2, \gamma_{\text{id}} - 1 \end{array} \right\rangle.$$

This simplifies to

$$H_{CR}^*(M) \cong \mathbb{Q}[u_1, u_2, \gamma] / \langle \gamma^2, u_1^3, u_2^3, u_1 u_2^2, u_1^2 u_2, \gamma u_1, \gamma u_2 \rangle.$$

Here,  $\deg(u_i) = 2, \deg(\gamma) = 4$ . We see that the orbifold Poincaré polynomial is

$$P_{\text{orb}}(t, M) = 1 + 2t^2 + 4t^4,$$

so the orbifold Euler characteristic is 7.

### Appendix A: Inertial Cohomology and Chen–Ruan Cohomology

In this section, we show that there is a natural equivalence between the inertial cohomology of a stably complex space  $Z$  from Sec. 3 and the orbifold cohomology of  $Z/T$  when  $T$  acts locally freely, i.e. that there exists a graded ring isomorphism

$$NH_T^{*,\diamond}(Z) \cong H_{CR}^*(Z/T). \tag{A.1}$$

Applying this isomorphism to the case when  $Z$  is a level set of the hyperkähler moment map on  $T^*\mathbb{C}^n$  and  $M = Z/T$  is an orbifold hypertoric variety completes the proof of Theorem 1.1. (There is a proof of a similar statement in [5], but here we drop their compactness assumption.) In addition, we show that the definition of the product structure for  $NH_T^{*,\diamond}(Z)$  is also equivalent to another description used in the algebraic-geometry literature (e.g. [4, 2]).

We first prove (A.1) as additive groups. We simplify the presentation in [3] to the case when the group involved is abelian, and  $X = Z/T$ . Let  $X_t := \{(p, t) : p \in Z/T, t \in G_p\}$ , where  $G_p$  is the local orbifold structure group at the point

$p \in Z/T$ . We assume for simplicity that  $X_t$  is connected (if not, take a direct sum over connected components). By definition,

$$H_{CR}^d(X) := \bigoplus_{t \in T} H^{d-2\sigma_t}(X_t), \tag{A.2}$$

where the degree shift  $\sigma_t$  is constant on connected components, and is defined below. Since in our case  $X = Z/T$  is a global quotient, each  $G_p$  is a subgroup of  $T$  and  $X_t = \{(zT, t) : zT \in Z/T, t \in \text{Stab}(z)\}$ . In other words,  $X_t = Z_t/T$ , where  $Z_t := \{(z, t) : z \in Z, t \in \text{Stab}(z)\}$  and the  $T$ -action is on the first coordinate. Notice that, for a given  $t \in T$ ,  $Z_t \cong Z^t := \{z \in Z : t \cdot z = z\}$ . Since  $T$  acts locally freely on  $Z$ , it certainly acts locally freely on  $Z^t$ . Therefore,  $H^*(Z^t/T) \cong H_T^*(Z^t)$  (with  $\mathbb{Q}$  coefficients) and

$$H_{CR}^d(Z/T) \cong \bigoplus_{t \in T} H_T^{d-2\sigma_t}(Z^t) \tag{A.3}$$

as additive groups. The right-hand side of (A.3) is exactly the definition of  $NH_T^{*,\diamond}(Z)$ , so we have proved the additive isomorphism (A.1).

We now prove that the isomorphism (A.1) also preserves the grading. The number  $\sigma_t$  appearing in (A.2) is obtained as follows; we assume  $X_t \neq \emptyset$ . At any point  $p \in X$ , let  $\rho_p : G_p \rightarrow GL(k, \mathbb{C})$  be a representation specifying a local model  $\mathbb{C}^k/G_p$  at  $p$ . Since  $G_p$  is abelian, the image of  $\rho_p$  is simultaneously diagonalizable; denote by  $\{a_{\lambda_j}(t)\}_{j=1}^k$  the logweights of the eigenvalues of  $\rho_p(t), t \in G_p$ . The sum

$$\sigma_t := \sum_{j=1}^k a_{\lambda_j}(t) \in \mathbb{Q} \tag{A.4}$$

is well-defined, constant on connected components of  $X_t$ , and gives the degree shift in [3] and (A.2).

We now show that this degree shift encoded by  $\sigma_t$  agrees with the degree shift in the definition of the grading for inertial cohomology in [5]. The local model  $\mathbb{C}^k/G_p$  can also be obtained by looking at the original  $T$ -space  $Z$ . Namely, given a lift  $z$  of the point  $p \in X$ ,  $G_p$  is exactly  $\text{Stab}(z) \subseteq T$  and the representation  $\rho_p$  above is given by the action of  $G_p$  on the normal bundle  $\nu(T \cdot z, Z)$  in  $T_z Z$ . Moreover, since  $t$  acts trivially on  $Z^t$ , the only nontrivial eigenvalues of  $\rho_p(t)$  are those which occur in the representation of  $\langle t \rangle$  on a further quotient  $\nu(Z^t, Z)$ . In particular one may conclude that the sum (A.4) equals the sum

$$\text{age}(t) := \sum_{\mathbb{C}_\lambda \subset \nu(Z^t, Z)} a_\lambda(t). \tag{A.5}$$

Even if  $X$  is not compact, the grading shift is well defined (as long as  $X$  is finite-dimensional). In particular, the normal bundle  $\nu(Z^t, Z)$  does not degenerate as it goes out to infinity. This shows that the gradings agree.

We have left to show that the isomorphism (A.1) preserves the ring structure. The products on both  $NH_T^{*,\diamond}(Z)$  and  $H_{CR}^*(X)$  are defined using the notion of an

obstruction bundle, so we begin by showing that the obstruction bundle of Definition 3.3, defined upstairs on  $Z$ , descends to the obstruction bundle of Chen and Ruan, defined on the quotient  $X = Z/T$ . In their original paper [3], the authors define these bundles over 3-twisted sectors; however, their construction can be greatly simplified in the case of a global quotient  $X = Z/T$ , so we restrict attention to this case below.

Chen and Ruan define their obstruction bundle using two ingredients; we describe each in turn. Consider a point  $[z] \in X = Z/T$ , and suppose that  $t_1, t_2 \in G_{[z]} \subseteq T$ . Let  $H := \langle t_1, t_2 \rangle$  be the finite subgroup they generate. Let

$$X_{(t_1, t_2, t_3)} := \{(p, t_1, t_2, t_3) : p \in X, t_1, t_2, t_3 \in G_p, t_1 t_2 t_3 = 1\}.$$

Then there is a smooth map  $e : X_{(t_1, t_2, t_3)} \rightarrow X$  projecting to the first term. Let  $e^*TX$  be the pullback of the tangent bundle; this is a complex  $H$ -equivariant orbundle over  $X_{(t_1, t_2, t_3)}$  and is the first ingredient in the Chen–Ruan definition of the obstruction bundle.

The second ingredient involves only the subgroup  $H$ . Let  $\Sigma = (\Sigma, t_1, t_2, t_3, H)$  be a proper smooth Galois  $H$  cover of  $\mathbb{P}^1$  branched over  $\{0, 1, \infty\}$  (for details see [4, Appendix]). The  $H$ -action on  $\Sigma$  induces an  $H$ -action on  $H^1(\Sigma, \mathcal{O}_\Sigma)$ , so we may define the topologically trivial  $H$ -equivariant bundle with fiber  $H^1(\Sigma, \mathcal{O}_\Sigma)$  over  $X_{(t_1, t_2, t_3)}$  of complex rank  $\text{genus}(\Sigma)$ , where the  $H$ -action is only on the fiber. We denote this bundle by  $H^1(\Sigma, \mathcal{O}_\Sigma)$ . Then the obstruction bundle of Chen and Ruan is given by the  $H$ -invariant part of the tensor product of these two bundles, i.e.

$$E := (H^1(\Sigma, \mathcal{O}_\Sigma) \otimes e^*TX)^H. \tag{A.6}$$

We now wish to show that the obstruction bundle of Definition 3.3 descends to (A.6). As a first step, observe that  $X_{(t_1, t_2, t_3)}$  is isomorphic to  $Z^H/T$ , so the base spaces of the two bundles certainly correspond. One reasonable way to lift the bundle might be to replace  $e^*TX$  with  $e^*TZ$  in the Chen–Ruan definition. However, this tangent bundle is not complex. Since a fiber  $(e^*TX)_{[z]}$  of the orbundle  $e^*TX$  can be constructed via  $T$ -equivalence classes in  $\nu(T \cdot z, Z)$ , a natural idea would be to split  $e^*TZ$  at any point  $z$  into the tangent directions along the orbits (which should not contribute), and its (complex) quotient bundle,  $\nu(T \cdot z, Z)$ . Alternatively, one can split  $e^*TZ$  into the tangent directions  $TZ^H$  along the fixed point set, and its (complex) quotient  $\nu(Z^H, Z)$ . In either case,  $TZ^H$  (or its quotient in  $\nu(T \cdot z, Z)$ ) does not contribute to the obstruction bundle, since

$$(H^1(\Sigma, \mathcal{O}_\Sigma) \otimes TZ^H)^H = H^1(\Sigma, \mathcal{O}_\Sigma)^H \otimes TZ^H = H^1(\mathbb{C}P^1, \mathcal{O}_{\mathbb{C}P^1}) \otimes TZ^H = 0. \tag{A.7}$$

Thus only the normal bundle  $\nu(Z^H, Z)$  contributes, and we see that

$$\tilde{E} := (H^1(\Sigma, \mathcal{O}_\Sigma) \otimes \nu(Z^H, Z))^H \rightarrow Z^H \tag{A.8}$$

quotients to  $E$ . Note that  $\nu * (Z^H, Z)$  is well-defined, even if  $Z^H$  is not compact.  $Z^H$  is a closed submanifold containing as a submanifold the orbit through  $z$ ; thus

$\nu(Z^H, Z)$  is a quotient of a local model on  $\nu(T \cdot z, Z)$  of the representation downstairs.

It remains to show that the only  $H$ -invariant subspaces of  $\nu(Z^H, Z)$  which contribute to (A.8) are the  $H$ -isotypic components  $\mathbb{C}_\lambda \subseteq I_\lambda \subseteq \nu(Z^H, Z)$  with  $a_\lambda(t_1) + a_\lambda(t_2) + a_\lambda(t_3) = 2$ . We analyze each piece  $\mathbb{C}_\lambda$  separately. We use Čech cohomology to compute with the  $H^1(\Sigma, \mathcal{O}_\Sigma)$ , so let  $\mathcal{U} = \{U_i\}_{i \in I}$  be an  $H$ -invariant open cover of  $\Sigma$ , i.e. for every  $\sigma \subseteq I$  there exists  $\tau$  such that  $h \cdot U_\sigma = U_\tau$ . We denote this by  $h \cdot \sigma = \tau$  for simplicity.

We claim that for  $z \in Z^H$ , the fiber  $(\mathbb{C}_\lambda \otimes H^1(\Sigma, \mathcal{O}_\Sigma))^H$  is isomorphic as a  $\mathbb{C}$ -vector space to  $H^1(\Sigma, \mathcal{L}_\lambda)$ , where  $\mathcal{L}_\lambda$  is the sheaf of  $H$ -invariant sections of the topologically trivial  $H$ -equivariant line bundle  $L_\lambda = \Sigma \times \mathbb{C}_\lambda$  over  $\Sigma$ . This can be seen at the level of cochains by the map

$$\begin{aligned} \phi : (\mathbb{C}_\lambda \otimes C^\ell(\mathcal{U}, \mathcal{O}_\Sigma))^H &\longrightarrow C^\ell(\mathcal{U}, \mathcal{L}_\lambda) \\ z \otimes s &\longmapsto zs. \end{aligned}$$

It is straightforward to check that  $\phi$  is well-defined and an isomorphism using the definition of the  $H$ -action on  $\mathbb{C}_\lambda \otimes C^\ell(\mathcal{U}, \mathcal{O}_\Sigma)$ , which can be written, for  $\sigma \subseteq I$ ,

$$(h \cdot (z \otimes s))|_\sigma = e^{2\pi i a_\lambda(h)} z \otimes s|_{h^{-1}\sigma}.$$

It may also be checked that  $\phi$  commutes with the Čech differential, so  $(\mathbb{C}_\lambda \otimes H^1(\Sigma, \mathcal{O}_\Sigma))^H \cong H^1(\Sigma, \mathcal{L}_\lambda)$ , as desired.

Furthermore, it is shown in [2] that  $H^1(\Sigma, \mathcal{L}_\lambda) \cong H^1(\mathbb{P}^1, \mathcal{O}(-a_\lambda(t_1) - a_\lambda(t_2) - a_\lambda(t_3)))$ . The latter is 1-dimensional exactly when the sum inside is  $-2$  and 0-dimensional otherwise, so  $(\mathbb{C}_\lambda \otimes H^1(\Sigma, \mathcal{O}_\Sigma))^H$  contributes nontrivially to  $\tilde{E}$  if and only if  $a_\lambda(t_1) + a_\lambda(t_2) + a_\lambda(t_3) = 2$ . As a bundle, each of these contributions is a line bundle over  $Z^H$  given as a sub-bundle of  $\nu(Z^H, Z)$ , since by construction  $H^1(\Sigma, \mathcal{O}_\Sigma)$  is the trivial bundle over  $Z^H$ . We conclude that

$$\tilde{E} \cong \sum_{\substack{I_\lambda \subseteq \nu(Z^H, Z) \\ a_\lambda(t_1) + a_\lambda(t_2) + a_\lambda(t_3) = 2}} I_\lambda,$$

where  $I_\lambda$  is the isotypic component of  $\nu(Z^H, Z)$  of weight  $\lambda$ .

Finally, under the isomorphism  $H_T^*(Z^H) \cong H^*(Z^H/T)$ , the equivariant Euler class  $e_T(\tilde{E})$  is mapped to the ordinary Euler class  $e(E) \in H^*(Z^H/T)$ . The  $\smile$ -product is then constructed to be identical to the definition given in [3]. We have proven (A.1), which we record as follows.

**Theorem A.1.** *The inertial cohomology  $NH_T^{*,\diamond}(Z)$  is isomorphic as a graded ring to the orbifold cohomology  $H_{CR}^*(M)$ .*

We now prove the correspondence of our definition of the obstruction bundle with a description in terms of right derived functors used in the algebraic geometry literature (e.g. [2, 4]). For this exercise, it is convenient to use the description in (A.8). In the algebraic-geometric context, the definition of the obstruction bundle (in the case of a global quotient by a locally free action) over  $Z^H$  is given as

$R^1\pi_*^H(\pi^*TZ|_{Z^H})$ , where  $\Sigma, H$  are as above,  $\pi : Z^H \times \Sigma \rightarrow Z^H$  is the projection,  $\pi_*^H$  is the functor “pushforward and take  $H$ -invariants”, and  $R^1\pi_*^H$  is its first right derived functor. By an argument similar to (A.7), only the normal bundle  $\nu(Z^H, Z)$  contributes nontrivially to this bundle, so  $R^1\pi_*^H(\pi^*TZ|_{Z^H}) = R^1\pi_*^H(\pi^*\nu(Z^H, Z))$ . We will work with this second description; in particular, we will show that in our (not necessarily algebraic) context, the right-hand side of this equation is equal to our bundle (A.8).

We begin by computing  $R^1\pi_*(\pi^*\nu(Z^H, Z))$ . The sheaf of sections of  $\pi^*\nu(Z^H, Z)$  is a  $\mathcal{O}_{Z^H \times \Sigma}$ -module, where  $\mathcal{O}_{Z^H \times \Sigma}$  is the sheaf of smooth functions on  $Z^H \times \Sigma$  that are holomorphic restricted to any fiber of  $\pi$ . By the push-pull formula,

$$R^1\pi_*(\pi^*\nu(Z^H, Z)) = \nu(Z^H, Z) \otimes R^1\pi_*(\mathcal{O}_{Z^H \times \Sigma}).$$

Moreover, the pushforward sheaf  $\pi_*(\mathcal{O}_{Z^H \times \Sigma})$  can be described as

$$\pi_*(\mathcal{O}_{Z^H \times \Sigma}) = \mathcal{O}_{Z^H} \otimes \Gamma(\Sigma, \mathcal{O}_\Sigma),$$

where here  $\mathcal{O}_{Z^H}$  is the sheaf of smooth functions on  $Z^H$  and  $\mathcal{O}_\Sigma$  is the (usual) sheaf of holomorphic functions on  $\Sigma$ . This implies that  $R^1\pi_*(\mathcal{O}_{Z^H \times \Sigma}) = \mathcal{O}_{Z^H} \otimes H^1(\Sigma, \mathcal{O}_\Sigma)$ , so we finally have

$$R^1\pi_*^H(\pi^*\nu(Z^H, Z)) = (\nu(Z^H, Z) \otimes H^1(\Sigma, \mathcal{O}_\Sigma))^H \rightarrow Z^H,$$

as desired.

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