ORBIFOLD COHOMOLOGY OF TORUS QUOTIENTS

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Abstract
We introduce the inertial cohomology ring $\mathrm{NH}_T^{*,\diamond}(Y)$ *of a stably almost complex manifold carrying an action of a torus T. We show that in the case where Y has a locally free action by T, the inertial cohomology ring is isomorphic to the Chen-Ruan orbifold cohomology ring $H^*_{\mathrm{CR}}(Y/T)$ (as defined in [CR]) of the quotient orbifold Y/T.*

For Y a compact Hamiltonian T-space, we extend to orbifold cohomology two techniques that are standard in ordinary cohomology. We show that $\mathrm{NH}_T^{*,\diamond}(Y)$ has a natural ring surjection onto $H^*_{\mathrm{CR}}(Y/\!/T)$, where $Y/\!/T$ is the symplectic reduction of Y by T at a regular value of the moment map. We extend to $\mathrm{NH}_T^{*,\diamond}(Y)$ the graphical Goresky-Kottwitz-MacPherson (GKM) calculus (as detailed in, e.g., [HHH]) and the kernel computations of [TW] and [G1], [G2].

We detail this technology in two examples: toric orbifolds and weight varieties, which are symplectic reductions of flag manifolds. The Chen-Ruan ring has been computed for toric orbifolds, with $\mathbb{Q}$-coefficients, in [BCS]; we reproduce their results over $\mathbb{Q}$ for all symplectic toric orbifolds obtained by reduction by a connected torus (though with different computational methods) and extend them to $\mathbb{Z}$-coefficients in certain cases, including weighted projective spaces.

Contents

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1. Introduction

In [CR], Chen and Ruan introduced orbifold cohomology groups along with a product structure as part of a program to understand orbifold string theory. This *Chen-Ruan orbifold cohomology ring** $H^*_{\mathrm{CR}}(X)$ is the degree 0 part of the quantum cohomology of the orbifold X (which, when X is a manifold, reduces to the ordinary cohomology) and, as such, one of its subtlest properties is the associativity of its product. It was originally conjectured that $H^*_{\mathrm{CR}}(X;\mathbb{C})$ with complex coefficients is isomorphic as a ring to the ordinary cohomology $H^*(\widetilde{X};\mathbb{C})$ of a crepant resolution $\widetilde{X}$ of X, when one exists (see, e.g., [CR], [BCS]). In this way, $H^*_{\mathrm{CR}}(X)$ should record data about some of the simplest kinds of singularities of blowdowns, namely, orbifold singularities. For example, "simple" singularities in codimension 2 are all orbifold singularities. These arise by blowing down ADE-diagrams of rational curves in a surface. Ruan's quantum minimal model conjecture in [R2], [R1] modifies this conjecture slightly and involves corrections from the quantum cohomology of $\widetilde{X}$.

Fantechi and Göttsche simplified the presentation of $H^*_{\mathrm{CR}}(X)$ in [FG], in the case where X is the global quotient of a complex manifold by a finite (possibly, nonabelian) group. In the algebraic category, Abramovich, Graber, and Vistoli [AGV] described a story analogous to Chen and Ruan's for Deligne-Mumford stacks. Borisov, Chen, and Smith [BCS] used the [AGV] prescription to describe explicitly the Chen-Ruan cohomology for toric Deligne-Mumford stacks.

The goal of this article is to simplify the presentation of the Chen-Ruan cohomology ring for those orbifolds that occur as a global quotient by an abelian compact Lie group. These orbifolds were already intensively studied by Atiyah in [A], where he essentially computed an index theorem for them using a Chern character map taking values in what we now recognize to be their Chen-Ruan cohomology groups. These did not have a general definition at the time.

Our interest in this family of orbifolds is due to their origin in the study of symplectic reductions of Hamiltonian T-spaces. Recall that a symplectic manifold (Y,ω) carrying an action of a torus T is a *Hamiltonian T-space* if there is an invariant map $\Phi: Y \to \mathfrak{t}^*$ from Y to the dual of the Lie algebra of T satisfying

$$d\langle \Phi, \xi\rangle = \iota_{V_\xi}\omega \tag{1.1}$$

for all $\xi \in \mathfrak{t}$, where V_ξ is the vector field on Y generated by ξ. Throughout this article, we assume that some component $\langle \Phi, \xi\rangle$ of the moment map Φ is proper and bounded

*Here, we use H^*_{CR} to denote the Chen-Ruan orbifold cohomology ring. In [CR], Chen and Ruan call this ring *orbifold cohomology* and denote it H^*_{orb}, but this name and notation have been used multiple times in the literature.

below. We call a Hamiltonian T-space whose moment map satisfies this condition a *proper Hamiltonian T-space*. The most important examples are smooth projective varieties Y carrying a linear T-action; the symplectic form is the Fubini-Study form from the ambient projective space, and properness follows from compactness.

It follows from (1.1) that for any regular value μ of the moment map, $\Phi^{-1}(\mu)$ is a submanifold of Y with a locally free T-action. In particular, any point in the level set has at most a finite stabilizer in T. The symplectic reduction

$$Y//T(\mu) := \Phi^{-1}(\mu)/T$$

is thus an orbifold. In particular, many (but not all) toric orbifolds may be obtained by symplectic reduction of manifolds.

Recall that for any T-space Y, the inclusion $i : Y^T \hookrightarrow Y$ of the fixed point set induces a map

$$i^* : H_T^*(Y) \hookrightarrow H_T^*(Y^T)$$

in equivariant cohomology. If Y has the property that i^* is *injective* (over $\mathbb{Q}$, $\mathbb{Z}$, etc.), then we call Y *equivariantly injective* (over $\mathbb{Q}$, $\mathbb{Z}$, etc.). We say that Y is *equivariantly formal* (with respect to its T-action) if the E_2-term of the Leray-Serre spectral sequence

$$Y \hookrightarrow Y \times_T ET \to BT$$

collapses, implying $H_T^*(Y) \cong H^*(Y) \otimes H_T^*(pt)$ as modules over $H_T^*(pt)$. Over $\mathbb{Q}$, equivariant formality implies equivariant injectivity (see [GS2]). In particular, proper Hamiltonian T-spaces are always equivariantly injective over $\mathbb{Q}$.

For any $g \in T$, let Y^g denote the fixed point set of the g-action on Y. We say that Y is *robustly equivariantly injective* if, for every $g \in T$, the T-invariant submanifold Y^g is equivariantly injective. When Y is a proper Hamiltonian space and $H^*(Y^T)$ is free, Y is equivariantly injective over $\mathbb{Z}$—even robustly equivariantly injective since Y^g is itself a proper Hamiltonian space for every g.

Note that not all equivariantly injective spaces are robustly so: a counterexample is S^1 acting on $\mathbb{RP}^2$ by the "rotation" induced from the rotation action on the double cover S^2. The points fixed by the element of order 2 form the set $\mathbb{RP}^1 \cup \{pt\}$, which is not equivariantly injective. We thank C. Allday [Al] and V. Puppe [Pu] for each discovering this example and sharing it with us.

Our main contribution to the study of Chen-Ruan cohomology is the definition of the *inertial cohomology** of a stably almost complex manifold Y and, in particular, of

*In the announcement [GHK] of these results, we used the term *preorbifold cohomology*. We believe that *inertial* is more suggestive, referring to the inertia orbifold, whose cohomology we are studying. We use the notation NH^* for i*n*ertial cohomology because IH^* is the standard notation for intersection cohomology.

a proper Hamiltonian T-space, denoted $\mathrm{NH}_T^{*,\diamond}(Y)$. The inertial cohomology is defined as an $H_T^*(pt)$-module (but not as a ring) by

$$\mathrm{NH}_T^{*,\diamond}(Y) := \bigoplus_{g\in T} H_T^*(Y^g).$$

The product structure and even the grading on $\mathrm{NH}_T^{*,\diamond}(Y)$ are rather complicated, and we leave their definition until Section 3. The grading * is by *real* numbers, and $^\diamond$ is by elements of T. The summand $H_T^*(Y^g)$ in the above is the $\mathrm{NH}_T^{*,g}(Y) = \bigoplus_{r\in\mathbb{R}} \mathrm{NH}_T^{r,g}(Y)$ part of $\mathrm{NH}_T^{*,\diamond}(Y)$.

The collection of restriction maps $H_T^*(Y^g) \to H_T^*(Y^T)$ produces a map

$$\mathrm{NH}_T^{*,\diamond}(Y) \longrightarrow \bigoplus_{g\in T} H_T^*(Y^T) \cong H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]. \tag{1.2}$$

In Section 2, we define a product $\star$ on the target space. In the case of Y robustly equivariantly injective, the map (1.2) is *injective*, and $\star$ pulls back to a product on $\mathrm{NH}_T^{*,\diamond}(Y)$. Most importantly, this product is easy to compute.

In Section 3, we introduce the product $\smile$ on $\mathrm{NH}_T^{*,\diamond}(Y)$ for any stably complex T-manifold Y. Its definition is of no surprise to anyone who has computed with Chen-Ruan cohomology, though it does not seem to have been formalized before in terms of equivariant cohomology; it is set up to make it easy to show that

$$\mathrm{NH}_T^{*,\diamond}(\mathscr{Z}) \cong H_{\mathrm{CR}}^*(\mathscr{Z}/T)$$

for any space $\mathscr{Z}$ with a locally free T-action.

We then prove the essential fact that in the robustly equivariantly injective case, the product $\star$ can be used to compute the product $\smile$. The virtue of $\star$ is that it is easy to compute with; for example, it is essentially automatic to show that it is graded and associative. On the other hand, we need the $\smile$-product to show that we indeed have a well-defined product on $\mathrm{NH}_T^{*,\diamond}(Y)$ rather than merely on $H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$. In addition, $\smile$ is better behaved from a functorial point of view, such as when restricting to a level set of the moment map of a proper Hamiltonian T-space.

Suppose that Y is a proper Hamiltonian T-space, and suppose that $\mathscr{Z}$ is the zero-level set $\Phi^{-1}(\mu)$. One of our main theorems (Theorem 6.4) states that there is a surjection of graded (in the first coordinate) rings

$$\mathrm{NH}_T^{*,\diamond}(Y) \twoheadrightarrow H_{\mathrm{CR}}^*\big(Y/\!/T(\mu)\big) \tag{1.3}$$

arising from a natural restriction map $\mathrm{NH}_T^{*,\diamond}(Y) \longrightarrow \mathrm{NH}_T^{*,\diamond}(\mathscr{Z})$. This follows from the work of Kirwan [K] and the fact that $\mathrm{NH}_T^{*,\diamond}(\mathscr{Z}) \cong H_{\mathrm{CR}}^*(\mathscr{Z}/T)$. Furthermore, the ring $\mathrm{NH}_T^{*,\diamond}(Y)$ is *easy* to compute: the required data are readily available from the symplectic point of view. The kernel of the map (1.3) may be computed using techniques introduced by Tolman and Weitsman [TW] and refined by Goldin in [G1]. Essentially, our definitions and theorems are generalizations to the Chen-Ruan cohomology of similar ones about the maps

$$H_T^*(Y) \hookrightarrow H_T^*(Y^T), \qquad H_T^*(Y) \twoheadrightarrow H^*(Y/\!/T),$$

which are familiar in Hamiltonian geometry. An easy observation is that in order to compute $H^*_{\mathrm{CR}}(Y/\!/T(\mu))$ for any particular value μ, one only needs a much smaller ring. Let $\Gamma_\mu \subset T$ be the subgroup generated by all finite stabilizers occurring in the T-action on $\Phi^{-1}(\mu)$. Then

$$\mathrm{NH}_T^{*,\Gamma_\mu}(Y) := \bigoplus_{g\in\Gamma_\mu} H_T^*(Y^g)$$

is a subring of $\mathrm{NH}_T^{*,\diamond}(Y)$ which also surjects onto $H^*_{\mathrm{CR}}(Y/\!/T(\mu))$. In particular, if Γ is the subgroup generated by all finite stabilizers occurring in the T-action on Y, then $\mathrm{NH}_T^{*,\Gamma}(Y)$ surjects onto $H^*_{\mathrm{CR}}(Y/\!/T(\mu))$ for every regular value μ.

We consider the whole ring $\mathrm{NH}_T^{*,\diamond}(Y)$ rather than just this subring because we think it is interesting in its own right; it also lends elegance to proofs and statements of results. While Γ is easily computed from T acting on Y, it is an unnecessary computational step in order to state the surjectivity result (1.3). In addition, there is no natural map on inertial cohomology given a homomorphism of groups $\Gamma_1 \to \Gamma_2$. For example, Proposition 5.1 does not hold if the finite stabilizers occurring on X are different from those occurring on Y. By allowing the group over which we take the direct sum to be as large as possible, we are able to obtain some results concerning the functoriality of $\mathrm{NH}_T^{*,\diamond}$.

The article is organized as follows. In Section 2, we define $\mathrm{NH}_T^{*,\diamond}(Y)$ as an $H_T^*(pt)$-module, the "restriction" map from $\mathrm{NH}_T^{*,\diamond}(Y)$ to $H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, and the product $\star$ on $H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$. We show that $\star$ is an associative product, and graded. At this stage, it is unclear that the image of the restriction map is closed under $\star$ (and this is delayed until Section 3), but if one accepts this as a black box, one can already begin computing examples. When Y is robustly equivariantly injective, the product $\star$ induces a product on $\mathrm{NH}_T^{*,\diamond}(Y)$.

In Section 3, we define the $\smile$-product on $\mathrm{NH}_T^{*,\diamond}(Y)$ for any stably complex manifold with a smooth T-action. This definition makes the grading $\diamond$ over elements of T obvious while obscuring the associativity and the grading by real numbers.

Our main theorem in Section 3, Theorem 3.6, is that the restriction map (1.2) is a ring homomorphism from $\smile$ to $\star$. In particular, the image is a subring, and when Y is robustly equivariantly injective, the $\star$-product can be used as a simple means of computing the ring $\mathrm{NH}_T^{*,\diamond}(Y)$. For example, the associativity and gradedness of $\star$

prove the same properties of $\smile$. In fact, $\smile$ has these properties even when Y is not robustly equivariantly injective.

In Section 4, we prove that the inertial cohomology (with the $\smile$-product) of a space $\mathscr{X}$ with a locally free T-action is isomorphic to the Chen-Ruan cohomology ring of the quotient orbifold; this is essentially a definition chase and was our motivation for $\smile$. We also show that for a stably almost complex manifold carrying a T-invariant function, the inclusion of a regular level set induces a well-defined map in inertial cohomology. As a corollary, we obtain surjectivity from the inertial cohomology ring of a proper Hamiltonian T-space to the Chen-Ruan cohomology ring of the symplectic reduction. This connection is elaborated on in Section 6. Finally, we spend significant effort in making these computations tenable. In Section 7, we give yet another description of the product; in Sections 8 and 9, we explore two important sets of examples, namely, weight varieties (symplectic reductions of coadjoint orbits) and symplectic toric orbifolds.

Since completing this work (announced in [GHK]), we received the preprint [CH], which also uses equivariant cohomology to study Kirwan surjectivity for abelian symplectic quotient orbifolds, though it does not address Kirwan injectivity. Chen and Hu's introduction of *twist factors* in [CH, Definition 2] into the de Rham models of ordinary and equivariant cohomology parallels closely our modified homomorphism in Section 7. A crucial ingredient in both our approach and [CH] is a simplification in the description of the obstruction bundle presented in Definition 3.1; we got it from [BCS, Proposition 6.3], whereas they rederive it in [CH, Proposition 1]. Then the work in our central theorem, Theorem 3.6, takes place in [CH, Proposition (Definition) 1, Proposition 1]. The fact that Chen and Hu's twist factors are cohomology classes rather than differential forms makes it appear more natural to work in cohomology from the start, as we do, which also allows us to work over $\mathbb{Z}$ rather than $\mathbb{R}$.

2. Inertial cohomology and the $\star$-product

Let Y be a stably almost complex T-space. For any group element $g \in T$, let $Y^g := \{y \in Y \mid g \cdot y = y\}$ denote the set of points fixed by g. Denote by $\mathrm{Stab}(y)$ the stabilizer of y in T. Since T is abelian, each Y^g is itself naturally a T-space.

Definition 2.1

The *inertial cohomology* of the space Y is given as an $H_T^*(pt)$-module by

$$\mathrm{NH}_T^{*,\diamond}(Y) := \bigoplus_{g\in T} H_T^*(Y^g),$$

where the sum indicates the $^\diamond$-grading; that is, $\mathrm{NH}_T^{*,g}(Y) = H_T^*(Y^g)$.

Notice that if Y is compact and T acts on Y locally freely, that is, if $\mathrm{Stab}(y)$ is finite for all $y \in Y$, then there are only finitely many nonzero terms in the sum in Definition 2.1. At the other extreme, if T has fixed points Y^T on Y, then every summand is nonzero.

Neither the ring structure nor the first grading * are the ones induced from each $H_T^*(Y^g)$. They are defined in Section 3 and depend on Y's stably almost complex structure, which Definition 3.1 does not.

For L, a 1-complex-dimensional representation of T with weight λ, and $g \in T$, the eigenvalue of g acting on L is $\exp(2\pi i\, a_\lambda(g))$, where $a_\lambda(g) \in [0, 1)$ is the *logweight* of g on L.

Since T preserves the stably almost complex structure on Y, any component F of the fixed point set Y^g is also stably almost complex, and the normal bundle νF to F in Y is an *actual* complex vector bundle. The torus T acts on νF and splits it into isotypic components

$$\nu F = \bigoplus_\lambda I_\lambda,$$

where the sum is over weights $\lambda \in \widehat{T}$. We denote the logweight of g on I_λ, restricted to any point $y \in F$, by $a_\lambda^F(g)$.

For each $g \in T$, there is an inclusion $Y^T \hookrightarrow Y^g$, inducing a map backward in T-equivariant cohomology. Note that for most g, this inclusion is equality. Put together, these give a *restriction* map

$$i_{\mathrm{NH}}^* : \mathrm{NH}_T^{*,\diamond}(Y) \longrightarrow \bigoplus_{g\in T} H_T^*(Y^T) \cong H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T],$$

where the isomorphism is as $H_T^*(pt)$-modules. If there are no fixed points, this map is zero. The most interesting case is when Y is robustly equivariantly injective, meaning that this map is injective.

For $b \in H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, F a component of Y^T, and $g \in T$, let $b|_{F,g} \in H_T^*(F)$ denote the *component* of b in the g-summand, restricted to the fixed point component F. Only finitely many of these components can be nonzero, and b can be reconstructed from them as

$$b = \sum_{F,g} (b|_{F,g}) \otimes 1g \;\in H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T],$$

where $b|_{F,g} \in H_T^*(F)$ lives in $H_T^*(Y^T)$ via the decomposition $H_T^*(Y^T) = \bigoplus_{F_i} H_T^*(F_i)$.

Definition 2.2

Let Y be a stably almost complex T-space. Let $b_1, b_2 \in H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$. The product $b_1 \star b_2$ is defined componentwise:

$$(b_1 \star b_2)|_{F,g} := \sum_{\substack{(g_1,g_2):\\ g_1g_2=g}} (b_1|_{F,g_1})(b_2|_{F,g_2}) \prod_{I_\lambda \subset \nu F} e(I_\lambda)^{a_\lambda^F(g_1)+a_\lambda^F(g_2)-a_\lambda^F(g_1g_2)}, \tag{2.1}$$

where $e(I_\lambda) \in H_T^*(F)$ is the equivariant Euler class of I_λ.

More generally, we have an n-ary product:

$$(b_1 \star \cdots \star b_n)|_{F,g} := \sum_{\substack{(g_1,\ldots,g_n):\\ \prod g_i=g}} \prod_i (b_i|_{F,g_i}) \prod_{I_\lambda \subset \nu F} e(I_\lambda)^{a_\lambda^F(g_1)+\cdots+a_\lambda^F(g_n)-a_\lambda^F(g_1g_2\cdots g_n)}. \tag{2.2}$$

Note that the exponent in (2.2) is an integer from zero to $n-1$ and, in fact, is the greatest integer $\lfloor a_\lambda^F(g_1)+\cdots+a_\lambda^F(g_n)\rfloor$. Since the sum is over

$$\Big\{(g_1,\ldots,g_n) : \prod g_i = g\Big\},$$

this product plainly respects the T grading from the second factor of $H_T^*(Y^T)\otimes_{\mathbb{Z}} \mathbb{Z}[T]$.

THEOREM 2.3
The 2*-fold product* $\star$ *is associative, making* $H_T^*(Y^T)\otimes_{\mathbb{Z}} \mathbb{Z}[T]$ *into a ring.*

Proof
Let $b_1, b_2, b_3 \in H_T^*(Y^T)\otimes_{\mathbb{Z}} \mathbb{Z}[T]$. We relate the 2-fold product to the 3-fold:

$$(b_1 \star b_2)\star b_3 = b_1 \star b_2 \star b_3 = b_1 \star (b_2 \star b_3).$$

More generally, the n-fold product can be built from the 2-fold with any parenthesization.

There are two multiplicative contributions to a $\star$-product: the components $b_i|_{F,g}$ and the equivariant Euler classes of the I_λ. For a given F and triple g_1, g_2, g_3 of group elements, the components give the same contribution:

$$(b_1|_{F,g_1}\cdot b_2|_{F,g_2})\cdot b_3|_{F,g_3} = b_1|_{F,g_1}\cdot b_2|_{F,g_2}\cdot b_3|_{F,g_3} = b_1|_{F,g_1}\cdot (b_2|_{F,g_2}\cdot b_3|_{F,g_3}).$$

To see that the exponents match on the equivariant Euler class of I_λ, we check

$$\begin{aligned}
&[a_\lambda^F(g_1)+a_\lambda^F(g_2)-a_\lambda^F(g_1g_2)] + \big[a_\lambda^F(g_1g_2)+a_\lambda^F(g_3)-a_\lambda^F\big((g_1g_2)g_3\big)\big]\\
&\quad = a_\lambda^F(g_1)+a_\lambda^F(g_2)+a_\lambda^F(g_3)-a_\lambda^F(g_1g_2g_3)\\
&\quad = [a_\lambda^F(g_2)+a_\lambda^F(g_3)-a_\lambda^F(g_2g_3)] + \big[a_\lambda^F(g_1)+a_\lambda^F(g_2g_3)-a_\lambda^F\big(g_1(g_2g_3)\big)\big],
\end{aligned}$$

which is plain. $\square$

Remark 2.4
If $b_1 \in H_T^*(Y^T) \otimes \mathbf{1}$, $b_2 \in H_T^*(Y^T) \otimes g$, where $\mathbf{1}$ is the identity element of T, then $b_1 \star b_2 \in H_T^*(Y^T) \otimes g$, and

$$\begin{aligned}(b_1 \star b_2)|_F &= (b_1|_{F,\mathbf{1}})(b_2|_{F,g}) \prod_{I_\lambda \subset \nu F} e(I_\lambda)^{a_\lambda^F(\mathbf{1}) + a_\lambda^F(g) - a_\lambda^F(g)} \\ &= (b_1|_{F,\mathbf{1}})(b_2|_{F,g}) \prod_{I_\lambda \subset \nu F} e(I_\lambda)^0 = (b_1|_{F,\mathbf{1}})(b_2|_{F,g})\end{aligned}$$

is the ordinary multiplication.

What is not clear at this point is that the image of the restriction map from $\mathrm{NH}_T^{*,\diamond}(Y)$ into $H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ is closed under $\star$. This follows from Theorem 3.6.

We now turn to the grading in $H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ which is built from logweights. In particular, it is not, in general, graded by integers, nor is it even graded by rationals (unlike Chen-Ruan cohomology).

Let $y \in Y^g$, and decompose $T_y Y = \bigoplus_j L_j$ under the g-action. The sum of the logweights of g on each of these lines is termed the *age* of g at y (see [R]). Since this number depends only on the connected component Z of y in Y^g, let

$$\mathrm{age}(Z, g) = \sum_j a_{\lambda_j}(g).$$

It is, in general, a real number. If g is of finite order n, then $\mathrm{age}(Z, g) \in (1/n)\mathbb{Z}$, but generically, the group element g need not be of finite order (see Remark 6.9).

Let $b \in H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ such that $b|_{Z,g}$ is zero except for one $g \in T$ and one component Z of Y^g. Further, assume that $b|_{Z,g} \in H_T^*(Z)$ is homogeneous. Then we assign

$$\deg(b) = \deg(b|_{Z,g}) + 2\,\mathrm{age}(Z, g).$$

The factor of 2 is the usual conversion factor $\dim_{\mathbb{R}} \mathbb{C}$; if one works with Chow rings rather than cohomology, one does not include it.

THEOREM 2.5
This definition of degree makes $(H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T], \star)$ into a graded algebra.

Proof
It is enough to check when b_1, b_2 each have only one nonvanishing component $b_1|_{F_1,g_1}$, $b_2|_{F_2,g_2}$, and hence degrees

$$\deg(b_i) = \deg(b_i|_{F_i,g_i}) + 2\,\mathrm{age}(F_i, g_i).$$

If $F_1 \neq F_2$, then $b_1 \star b_2 = 0$, and there is nothing to prove. Otherwise, $b_1 \star b_2$ has only one nonvanishing component,

$$(b_1 \star b_2)|_{F,g_1g_2} = b_1|_{F,g_1} b_2|_{F,g_2} \prod_{I_\lambda \subset \nu F} e(I_\lambda)^{a_\lambda^F(g_1)+a_\lambda^F(g_2)-a_\lambda^F(g_1g_2)},$$

where $F = F_1 = F_2$. The degree that we assign to this is

$$\begin{aligned}
&\deg(b_1 \star b_2) \\
&\quad = \Big(\deg b_1|_{F,g_1} + \deg b_2|_{F,g_2} \\
&\qquad + \sum_{I_\lambda \subset \nu F} \big(a_\lambda^F(g_1) + a_\lambda^F(g_2) - a_\lambda^F(g_1g_2)\big)(2 \dim_{\mathbb{C}} I_\lambda)\Big) \\
&\qquad + 2\,\mathrm{age}(F, g_1g_2).
\end{aligned}$$

Canceling the $b_i|_{F,g_i}$-contributions and the factor of 2, our remaining task is therefore to show that

$$\begin{aligned}
&\mathrm{age}(F, g_1) + \mathrm{age}(F, g_2) \\
&\quad = \mathrm{age}(F, g_1g_2) + \sum_{I_\lambda \subset \nu F} \big(a_\lambda^F(g_1) + a_\lambda^F(g_2) - a_\lambda^F(g_1g_2)\big) \dim_{\mathbb{C}} I_\lambda.
\end{aligned}$$

This follows from three applications of the formula

$$\mathrm{age}(F, g) = \sum_{I_\lambda \subset \nu F} a_\lambda^F(g) \dim_{\mathbb{C}} I_\lambda,$$

which is just a resummation of the definition. □

Our point of view is that the grading is not of fundamental importance—it happens to be preserved by the multiplication, so we record it as an extra tool for studying this ring. In Section 7, we see what is perhaps the best motivation for this grading.

3. Inertial cohomology and the $\smile$-product

The definition of the $\star$-product on $H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ renders the ring $\mathrm{NH}_T^{*,\diamond}(Y)$ straightforward to compute when Y is robustly equivariantly injective, as is shown in the examples in Sections 7 and 9. It also has the advantage that associativity is easy to prove. However, its limits are easy to see: if there are no fixed points, for example, $H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ is zero; we have no proof yet that $i_{\mathrm{NH}}^*(\mathrm{NH}_T^{*,\diamond}(Y))$ is a subring of $H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$; even assuming that, when Y is not robustly equivariantly injective, the $\star$-product does not let us define a product on the source $\mathrm{NH}_T^{*,\diamond}(Y)$ but only on its image inside $H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$. For these reasons, we present in this section a product $\smile$ directly on $\mathrm{NH}_T^{*,\diamond}(Y)$ for any stably almost complex manifold Y, and we show that i_{NH}^* is a ring homomorphism. The $\smile$-product has its roots in the original paper by

Chen and Ruan [CR], but it is defined using the global group action and the language of equivariant cohomology.

Let Y be a stably almost complex manifold with a smooth T-action respecting the stably almost complex structure. Note that this implies that each normal bundle $\nu(Y^{g_1,g_2} \subset Y^{g_1})$ is a complex vector bundle (not just stably so) over Y^{g_1,g_2} for every choice of $g_1, g_2 \in T$, where $Y^{g_1,g_2} = (Y^{g_1})^{g_2} = (Y^{g_2})^{g_1}$. The definition of the product $\smile$ requires the introduction of a new space and a vector bundle over each of its connected components. Let

$$\widetilde{Y} := \coprod_{g_1,g_2 \in T} Y^{g_1,g_2}.$$

For any connected component Z of Y^{g_1,g_2}, the group $\langle g_1, g_2 \rangle$ generated by g_1 and g_2 acts on the complex vector bundle νZ, the normal bundle to Z in Y, fixing Z itself. Thus, as a representation of $\langle g_1, g_2 \rangle$, νZ breaks up into isotypic components

$$\nu Z = \bigoplus_{\lambda \in \widehat{\langle g_1, g_2 \rangle}} I_\lambda,$$

where I_λ is the bundle over Z on which $\langle g_1, g_2 \rangle$ acts with representation given by λ.

Definition 3.1
For each connected component Z of Y^{g_1,g_2} in $\widetilde{Y}$, let $E|_Z$ be the vector bundle over Z given by

$$E|_Z = \bigoplus_{a_\lambda(g_1)+a_\lambda(g_2)+a_\lambda(g_3)=2} I_\lambda,$$

where $g_3 := (g_1 g_2)^{-1}$. The *obstruction bundle* E is given by the union of $E|_Z$ over all connected components Z in $\widetilde{Y}$.

This sum of three logweights is reminiscent of, but *not* like, age: one calculates the age of a group element by summing over lines, whereas this functional is calculated for a line by summing over group elements.

Note that the dimensions of the fibers of E may differ on different connected components.

Remark 3.2
Each component Z is T-invariant, and hence $E|_Z \to Z$ is a T-equivariant bundle. Thus there is a well-defined (inhomogeneous) equivariant Euler class ϵ of E: for every component Z, let ϵ restricted to Z be the equivariant Euler class of $E|_Z$. The class ϵ is also called the *virtual fundamental class* of $\widetilde{Y}$.

Consider the three inclusion maps given by

$$e_1 : Y^{g_1,g_2} \hookrightarrow Y^{g_1},$$
$$e_2 : Y^{g_1,g_2} \hookrightarrow Y^{g_2},$$
$$\overline{e}_3 : Y^{g_1,g_2} \hookrightarrow Y^{g_1g_2}.$$

The notation is explained in Section 4. The maps $e_1, e_2, \overline{e}_3$ clearly extend to maps on $\widetilde{Y}$. They therefore induce the pullbacks

$$e_1^*, e_2^* : \mathrm{NH}_T^{*,\diamond}(Y) \longrightarrow \bigoplus_{g_1,g_2 \in T} H_T^*(Y^{g_1,g_2})$$

and the pushforward map

$$(\overline{e}_3)_* : \bigoplus_{g_1,g_2 \in T} H_T^*(Y^{g_1,g_2}) \longrightarrow \mathrm{NH}_T^{*,\diamond}(Y).$$

Definition 3.3
For $b_1, b_2 \in \mathrm{NH}_T^{*,\diamond}(Y)$, define

$$b_1 \smile b_2 := (\overline{e}_3)_* \left(e_1^*(b_1) \cdot e_2^*(b_2) \cdot \epsilon\right),$$

where ϵ is the virtual fundamental class of $\widetilde{Y}$ and the product occurring in the right-hand side is that in the equivariant cohomology of each piece Y^{g_1,g_2} of $\widetilde{Y}$.

Remark 3.4
While Y^{g_1,g_2} and $Y^{g_1g_2}$ may be noncompact, $\overline{e}_3$ is still a closed embedding, so the pushforward is well defined.

Remark 3.5
If $b_1 \in \mathrm{NH}_T^{*,g_1}(Y)$ and $b_2 \in \mathrm{NH}_T^{*,g_2}(Y)$, then $e_1^*(b_1)$, $e_2^*(b_2)$, and their product $e_1^*(b_1) \cdot e_2^*(b_2)$ live in $H_T^*(Y^{g_1,g_2})$. After we multiply by ϵ, the pushforward map $(\overline{e}_3)_*$ sends this class to $H_T^*(Y^{g_1g_2})$, which implies that $b_1 \smile b_2 \in \mathrm{NH}_T^{*,g_1g_2}(Y)$.

We also define a real-valued grading on $\mathrm{NH}_T^{*,\diamond}(Y)$. Recall that

$$\mathrm{NH}_T^{*,g}(Y) = H_T^*(Y^g) = \bigoplus_Z H_T^*(Z),$$

where Z varies over the connected components of Y^g. We shift the degree on the $H_T^*(Z)$-summand by (twice) the age of g on any tangent space T_zY, $z \in Z$. In particular, we usually do *not* shift all of $H_T^*(Y^g)$ by the same amount.

At this point we have not shown that $\smile$ is associative, so the main theorem of this section is phrased in terms of “not necessarily associative rings.”

THEOREM 3.6
Let Y be a stably almost complex manifold with T-action, and let $b_1, b_2 \in \mathrm{NH}_T^{*,\diamond}(Y)$. Let $i_{\mathrm{NH}}^* : \mathrm{NH}_T^{*,\diamond}(Y) \to H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ be the restriction map. Then

$$i_{\mathrm{NH}}^*(b_1 \smile b_2) = i_{\mathrm{NH}}^*(b_1) \star i_{\mathrm{NH}}^*(b_2);$$

that is, i_{NH}^* is a homomorphism of not necessarily associative rings. Moreover, it preserves the real-valued grading.

Proof
Suppose that $b_1 \in \mathrm{NH}_T^{*,g_1}(Y)$ and $b_2 \in \mathrm{NH}_T^{*,g_2}(Y)$. It is enough to prove it for this case. Our goal is to prove

$$i_{\mathrm{NH}}^*(b_1 \smile b_2)\big|_{F,g} = \big(i_{\mathrm{NH}}^*(b_1) \star i_{\mathrm{NH}}^*(b_2)\big)\big|_{F,g}$$

for every (F, g)-component of either side. It is easy to see that both sides vanish unless $g = g_1 g_2$.

Let F be a connected component of the fixed point set Y^T, which therefore includes T equivariantly into Y^{g_1,g_2}, which in turn includes into each of Y^{g_1}, Y^{g_2}, $Y^{g_1 g_2}$. Call this first map $i_F : F \to Y^{g_1,g_2}$, the others already having the names $e_1, e_2, \overline{e}_3$.

We start with the left-hand side. Let Z denote the connected component of Y^{g_1,g_2} containing F, let ϵ_Z denote the Euler class of the obstruction bundle on Z, and let f_Z denote the equivariant Euler class of Z's normal bundle inside $Y^{g_1 g_2}$. Then

$$\begin{aligned}
&i_{\mathrm{NH}}^*(b_1 \smile b_2)|_{F,g_1 g_2} \\
&\quad = (\overline{e}_3 \circ i_F)^*\big((\overline{e}_3)_*(e_1^*(b_1) \cdot e_2^*(b_2) \cdot \epsilon_Z)\big) && \text{(by the definition of } \smile) \\
&\quad = i_F^* \overline{e}_3^*\big((\overline{e}_3)_*(e_1^*(b_1) \cdot e_2^*(b_2) \cdot \epsilon_Z)\big) \\
&\quad = i_F^*\big(e_1^*(b_1) \cdot e_2^*(b_2) \cdot \epsilon_Z \cdot \overline{e}_3^*((\overline{e}_3)_* 1)\big) && \text{(by the pull-push formula)} \\
&\quad = i_F^* e_1^*(b_1) \cdot i_F^* e_2^*(b_2) \cdot i_F^* \epsilon_Z \cdot i_F^* f_Z && \text{(by the definition of Euler class).}
\end{aligned}$$

Now, we compare it to the right-hand side:

$$\begin{aligned}
&\big(i_{\mathrm{NH}}^*(b_1) \star i_{\mathrm{NH}}^*(b_2)\big)\big|_{F,g_1 g_2} \\
&\quad = i_{\mathrm{NH}}^*(b_1)|_{F,g_1} \cdot i_{\mathrm{NH}}^*(b_2)|_{F,g_2} \cdot \prod_{I_\lambda \subset \nu F} e(I_\lambda)^{a_\lambda^F(g_1) + a_\lambda^F(g_2) - a_\lambda^F(g_1 g_2)} \\
&\quad = i_F^* e_1^*(b_1) \cdot i_F^* e_2^*(b_2) \cdot \prod_{I_\lambda \subset \nu F} e(I_\lambda)^{a_\lambda^F(g_1) + a_\lambda^F(g_2) - a_\lambda^F(g_1 g_2)}.
\end{aligned}$$

Our goal is thus to show that

$$i_F^* \epsilon_Z \cdot i_F^* f_Z = \prod_{I_\lambda \subset \nu F} e(I_\lambda)^{a_\lambda^F(g_1)+a_\lambda^F(g_2)-a_\lambda^F(g_1 g_2)}.$$

In fact, we show by case analysis that for each $\lambda \in \widehat{\langle g_1, g_2 \rangle}$, the equivariant Euler class of the bundle I_λ over F shows up to the same power on the left-hand and right-hand sides. Let

$$\varepsilon(\lambda) = \begin{cases} 1 & \text{if } a_\lambda^F(g_1) + a_\lambda^F(g_2) + a_\lambda^F(g_3) = 2, \text{ where } g_3 = (g_1 g_2)^{-1}, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$f(\lambda) = \begin{cases} 1 & \text{if } I_\lambda \subset \nu(Y^{g_1, g_2} \subset Y^{g_1 g_2}), \\ 0 & \text{otherwise.} \end{cases}$$

So $i_F^* \epsilon_Z = \prod_\lambda e(I_\lambda)^{\varepsilon(\lambda)}$ and $i_F^* f_Z = \prod_\lambda e(I_\lambda)^{f(\lambda)}$, and it remains to check that

$$\varepsilon(\lambda) + f(\lambda) = a_\lambda^F(g_1) + a_\lambda^F(g_2) - a_\lambda^F(g_1 g_2). \tag{3.1}$$

The main principle in the following case analysis is that $a_\lambda^F(g_1)+a_\lambda^F(g_2)-a_\lambda^F(g_1 g_2)$ is either 0 or 1, not some arbitrary real number, and likewise, $a_\lambda^F(g_1)+a_\lambda^F(g_2)+a_\lambda^F(g_3)$ is either 0, 1, or 2.

Assume first that $a_\lambda^F(g_1 g_2) = 0$, meaning that $g_1 g_2$ acts trivially on I_λ. Then $a_\lambda^F(g_3) = 0$; hence $\varepsilon(\lambda) = 0$, and also, $I_\lambda \leq TY^{g_1, g_2}$. So the equation that we seek is $f(\lambda) = a_\lambda^F(g_1)+a_\lambda^F(g_2)$, where both sides are either 0 or 1. Now, $f(\lambda) = 0$ if and only if g_1 and g_2 each act trivially on I_λ, which occurs exactly when $a_\lambda^F(g_1) = a_\lambda^F(g_2) = 0$.

On the other hand, assume that $a_\lambda^F(g_1 g_2) \neq 0$. Then $I_\lambda \not\leq TY^{g_1 g_2}$; hence $f(\lambda) = 0$. So the equation we want now is $\varepsilon(\lambda) = a_\lambda^F(g_1) + a_\lambda^F(g_2) - a_\lambda^F(g_1 g_2)$. The right-hand side is 1 if and only if $a_\lambda^F(g_1) + a_\lambda^F(g_2) > 1$, which happens precisely when $a_\lambda^F(g_1) + a_\lambda^F(g_2) + a_\lambda^F(g_3) = 2$. This occurs if and only if $\varepsilon(\lambda) = 1$. Thus, in either case, we have shown that (3.1) holds.

It is trivial to check that the grading is respected, essentially because both gradings are defined using ages. In particular, the proof does not require splitting into cases. □

If Y is robustly equivariantly injective, then i_{NH}^* is an injection. This follows from the injection $H_T^*(Y^g) \to H_T^*(Y^T)$ for each $g \in T$ and from Theorem 3.6.

COROLLARY 3.7

If Y is a robustly equivariantly injective T-space, then the $\smile$-product on $\mathrm{NH}_T^{*,\diamond}(Y)$ *and the bigrading can be inferred (using i_{NH}^*) from the $\star$-product and the bigrading on $H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$.*

One way of reading Theorem 3.6 is that the cases that occur in computing the $\smile$-product—which I_λ contribute to the obstruction bundle versus which I_λ are in the normal bundle and hence contribute to the $\overline{e}_3$-pushforward—"cancel" one another to some extent when taken together, making the $\star$-product simpler than either one considered individually.

In particular, the easy proofs of associativity and gradedness for $\star$ imply the same for $\smile$, in the robustly equivariantly injective case. In fact, these properties hold regardless.

THEOREM 3.8

The ring $(\mathrm{NH}_T^{*,\diamond}(Y), \smile)$ *is bigraded and associative for any stably almost complex T-manifold Y.*

Proof

In the case where Y is robustly equivariantly injective, we use Corollary 3.7. More generally, the proof can be accomplished by a case-by-case analysis (parallel to that in Theorem 3.6) of the bundles $E|_{Y^{g_1,g_2}}$ and the normal bundles Y^{g_1,g_2} in $Y^{g_1 g_2}$, where these bundles vary over each pair (g_1, g_2) and over each connected component of Y^{g_1,g_2}. □

4. Relation to orbifold cohomology of torus quotients

Our goal in this section is to perform the following definition chase.

THEOREM 4.1

Let T act on the compact, stably almost complex manifold $\mathscr{Z}$ locally freely, and let $X = \mathscr{Z}/T$ be the quotient orbifold. Then

$$\mathrm{NH}_T^{*,\diamond}(\mathscr{Z}) = H_{\mathrm{CR}}^*(X),$$

where $H_{\mathrm{CR}}^*(X)$ is as defined in [CR].

It becomes clear, as we recapitulate Chen and Ruan's definition of H_{CR}^*, that we have set up our definition of $\mathrm{NH}_T^{*,\diamond}$ in order to make this tautological.

Some of the difficulty in Chen and Ruan's definition arises from the technicalities of dealing with general orbifolds, but this can be sidestepped in the case of a global quotient. At one point, we make use of a different simplification of their definition which is found in [BCS].

Proof
Define

$$\widetilde{\mathscr{Z}} = \left\{(z, g) \,\middle|\, z \in \mathscr{Z}, g \in T, g \cdot z = z\right\} \subseteq \mathscr{Z} \times T.$$

For each $z \in \mathscr{Z}$, the stabilizer group is closed. By the local freeness, each stabilizer group is discrete and hence finite. By the compactness of $\mathscr{Z}$, only finitely many stabilizer groups occur up to conjugacy—but since T is abelian, we can omit "up to conjugacy." Hence only a finite set of g arises this way. Therefore the cohomology $H_T^*(\widetilde{\mathscr{Z}})$ is a direct sum over these $g \in T$, and in fact, this direct sum is exactly our definition of the inertial cohomology:

$$H_T^*(\widetilde{\mathscr{Z}}) = \mathrm{NH}_T^{*,\diamond}(\mathscr{Z}).$$

Since T is abelian, $(z, g) \in \widetilde{\mathscr{Z}}$ implies $(tz, g) \in \widetilde{\mathscr{Z}}$ for all $t \in T$, so we can form the quotient by this T-action. Following [CR, Section 3.1], we call this quotient orbifold $\widetilde{X} \subseteq X \times T$. Note that when X is a manifold, $\widetilde{X} = X \times \{1\}$.

We define

$$H_{\mathrm{CR}}^*(X) := H^*(\widetilde{X}) \quad \text{as a group.}$$

If we work with real coefficients as in [CR], then the right-hand side is just the ordinary cohomology of the underlying topological space. However, we generally prefer to use the integer cohomology of the classifying space of the orbifold (as in, e.g., [H]). In the case at hand, it means

$$H^*(\widetilde{X}) = H_T^*(\widetilde{\mathscr{Z}}) = \mathrm{NH}_T^{*,\diamond}(\mathscr{Z}).$$

We are not done, though, as we still have to consider the ring structure (and the grading, but we leave that to the reader).

To define the ring structure, we use $\widetilde{X}_3^\circ = \{(x, g_1, g_2, g_3) \mid g_i \in T_x, g_1g_2g_3 = 1\}$, a subset of

$$\widetilde{X}_3 = \left\{(x, g_1, g_2, g_3) \,\middle|\, g_i \in T_x\right\},$$

called the *3-multi-sector* in [CR, Section 4.1]. There are three natural maps

$$e_i : \widetilde{X}_3 \to \widetilde{X}$$

defined by $e_i(x, g_1, g_2, g_3) = (x, g_i)$ for $i = 1, 2, 3$. For each map e_i, we define $\overline{e_i} : \widetilde{X}_3 \to \widetilde{X}$ by $\overline{e_i}(x, g_1, g_2, g_3) = (x, g_i^{-1})$.

The definition of the obstruction bundle in [CR] is very complicated, but it is simplified a great deal in [BCS]. Let $\widetilde{\mathscr{Z}}_3$ be the 3-multi-sector, and let $\widetilde{\mathscr{Z}}_3^\circ = \{(z, g_1, g_2, g_3) \mid z \in Z^{g_1,g_2},\ g_1g_2g_3 = 1\}$. Let F be a connected component of $\widetilde{\mathscr{Z}}_3^\circ$, and let F' be its projection to $\mathscr{Z}$, considered as a component of the space $\widetilde{Y}$ from Section 3. While F does not have an almost complex structure, the construction of the *obstruction bundle* $E|_F$ over F of [CR, Section 4.1] works here as well since the normal bundle to any component of Z^{g_1,g_2} in Z is almost complex, and the tangent directions to Z^{g_1,g_2} do not contribute to the obstruction bundle. In [BCS, Proposition 6.3], Borisov, Chen, and Smith prove that this obstruction bundle is the quotient by T of the vector bundle F' from Definition 3.1.* In [CR], Chen and Ruan consider the Euler class of this orbibundle as an element of $H^*(F'/T)$. In the case of a global quotient orbifold F'/T, such an Euler class can instead be computed as the equivariant Euler class of the vector bundle, living in the isomorphic group $H_T^*(F')$. This is exactly what we used in the definition of $\smile$.

Let ϵ denote the sum of these Euler classes over all components, either in $H_{\mathrm{CR}}^*(X)$ or $\mathrm{NH}_T^{*,\diamond}(\mathscr{X})$. Then both definitions give the product of α and β as $\overline{e_3}_*(e_1^*(\alpha)\cdot e_2^*(\beta)\cdot\epsilon)$. □

5. Functoriality of inertial cohomology

Inertial cohomology is very far from being an equivariant cohomology theory for many of the same reasons that Chen-Ruan cohomology and quantum cohomology fail to be properly functorial as cohomology theories. The inertial cohomology *groups* are functorial: any T-equivariant map $f : X \to Y$ restricts to a T-equivariant map $f^t : X^t \to Y^t$ on the fixed sets by $t \in T$ and hence to a map $(f^t)^* : \mathrm{NH}_T^{*,t}(Y) \to \mathrm{NH}_T^{*,t}(X)$ backward on each summand.

Since the *rings* depend on the stably almost complex structures—and, more specifically, the honest complex structures on normal bundles to fixed point sets—we use conditions on these to guarantee that this map f^* is a ring homomorphism. While our conditions are extremely restrictive, we have two natural instances in which they are satisfied, one treated in Corollary 5.2 and one in Theorem 6.4.

PROPOSITION 5.1

Let $i : X \hookrightarrow Y$ be a T-equivariant inclusion so that Y is stably almost complex and the normal bundle to X in Y is trivialized. Then X is naturally stably almost complex.

Assume also that X is transverse to any Y^t, $t \in T$. Then the restriction map $i^* : \mathrm{NH}_T^{*,\diamond}(Y) \to \mathrm{NH}_T^{*,\diamond}(X)$ is a ring homomorphism.

Proof

The trivialization of the normal bundle gives an isomorphism between stabilizations of the tangent bundle of X and the restriction of the tangent bundle of Y. This proves the first claim.

*While the setting of [BCS] is the toric case, the calculation in [BCS, Proposition 6.3] works in general. Obstruction theory enters [CR] as the H^1 of a vector bundle constructed from the normal bundle to F. The subbundle from Definition 3.1 was selected out by asking that the sum $m = a_\lambda(g_1) + a_\lambda(g_2) + a_\lambda(g_3)$, a priori 0, 1, or 2, actually be 2. The link provided in [BCS, Proposition 6.3] between the two of these is to compute $H^1(\mathscr{O}(-m))$ over $\mathbb{C}P^1$, which vanishes unless the sum is 2.

The transversality guarantees that for each normal bundle $\nu(X^t \subseteq X)$, respectively, $\nu(X^{s,t} \subseteq X^{st})$, is the restriction of the corresponding normal bundle $\nu(Y^t \subseteq Y)$, respectively, $\nu(Y^{s,t} \in Y^{st})$, with the same logweights, and a simple calculation with these logweights shows that the product is the same. □

COROLLARY 5.2
Let Y be a stably almost complex T-space, and let X be the union of separated T-invariant tubular neighborhoods of the components of Y^T. Then the obvious isomorphism of groups

$$\mathrm{NH}_T^{*,\diamond}(X) \cong H_T^*(Y^T) \otimes_{\mathbb{Z}} \mathbb{Z}[T]$$

corresponds the $\smile$-product to the $\star$-product. The ring homomorphism i^* : $\mathrm{NH}_T^{*,\diamond}(Y) \to \mathrm{NH}_T^{*,\diamond}(X)$ composed with this isomorphism is the ring homomorphism i^*_{NH} from Section 3.

Proof
This is hardly more than a restatement of the definitions of $\mathrm{NH}_T^{*,\diamond}(X)$ and i^*_{NH}. To apply Proposition 5.1, we note that the normal bundle to X in Y is zero-dimensional and hence trivialized. □

Of course, the best case is that Y is robustly equivariantly injective, which is exactly the statement that this restriction map is an inclusion.

One way to think about Corollary 5.2 is the following. The ordinary restriction map in equivariant cohomology is usually thought of as going from $H_T^*(Y)$ to $H_T^*(Y^T)$, but it could equally well go to $H_T^*(X)$ since X equivariantly deformation retracts to Y^T. In the setting of inertial cohomology, by contrast, X is better than Y^T through being big enough to carry the geometric information with which we define the ring structure. Alternately, we can feed this information in by hand, which is how we defined the $\star$-product. In fact, the idea of replacing Y^T by the tubular neighborhood X has already shown up in the theory of noncompact Hamiltonian cobordism in [GGK].

6. Surjectivity for symplectic torus quotients

In this section, we relate the inertial cohomology of a Hamiltonian T-space Y to the Chen-Ruan cohomology of the symplectic reduction. Recall that the equivariant cohomology of Y surjects onto the ordinary cohomology of the reduced space $Y/\!/T$. Our first goal is the analogue for inertial cohomology, Theorem 6.4, showing that inertial cohomology surjects as a ring onto the Chen-Ruan cohomology of the reduced space. Our second goal is to compute the kernel of this map. With it, one can express $H^*_{\mathrm{CR}}(Y/\!/T)$ by computing $\mathrm{NH}_T^{*,\diamond}(Y)$ and quotienting by the kernel of a natural map, which we describe in Corollary 6.7. Indeed, there is a finitely generated subring $\mathrm{NH}_T^{*,\Gamma}(Y)$ of $\mathrm{NH}_T^{*,\diamond}(Y)$ which is sufficiently large to surject onto $H^*_{\mathrm{CR}}(Y/\!/T)$.

Suppose that Y is a Hamiltonian T-space with moment map $\Phi : Y \to \mathfrak{t}^*$. If zero is a regular value of Φ, then T acts locally freely on the level set $\Phi^{-1}(0)$, and we define the *symplectic reduction* at zero to be $Y/\!/T := \Phi^{-1}(0)/T$. An easy generalization of Marsden and Weinstein's result [MW, Theorem 5.16] implies that this is a *symplectic* orbifold. Kirwan used a variant of Morse theory to relate the equivariant topology of Y to the topology of $Y/\!/T$.

THEOREM 6.1 ([K, Corollary 5.17])
Let Y be a proper Hamiltonian T-space with moment map $\Phi : Y \to \mathfrak{t}^*$. Suppose that zero is a regular value of Φ, and suppose that M^T has only finitely many connected components. Then the inclusion $\Phi^{-1}(0) \hookrightarrow Y$ induces

$$\kappa \ : \ H^*_T(Y;\mathbb{Q}) \twoheadrightarrow H^*_T\big(\Phi^{-1}(0);\mathbb{Q}\big) \cong H^*(Y/\!/T;\mathbb{Q}), \tag{6.1}$$

a surjection in equivariant cohomology. The map κ is called the Kirwan map.

Remark 6.2
The fact that zero is a regular value of Φ implies that T acts locally freely on $\Phi^{-1}(0)$. This implies the isomorphism on the right-hand side of (6.1).

Remark 6.3
Theorem 6.1 does hold for other coefficient rings under additional hypotheses. A sufficient condition is that the Atiyah-Bott principle apply with the given coefficients for each critical set of $\|\Phi\|^2$. For circle actions, this is merely an additional hypothesis on the topology of the fixed point components. For the action of a torus T, this becomes a hypothesis on the K-equivariant cohomology of a critical set C, where $K \subseteq T$ is the largest subtorus acting locally freely on C. To apply the Atiyah-Bott principle over $\mathbb{Z}$, for example, it is sufficient to assume that $H^*_K(C;\mathbb{Z})$ is torsion-free. For further details, see [AB] and [TW].

Now we turn our attention to the relationship between the inertial cohomology of Y and the Chen-Ruan cohomology of the reduced space. As we have assumed that zero is a regular value of Φ, the action of T on the level set $\Phi^{-1}(0)$ is locally free. In particular, this implies that the symplectic reduction is naturally an orbifold.

THEOREM 6.4
Let Y be a proper Hamiltonian T-space with moment map $\Phi : Y \to \mathfrak{t}^*$. Suppose that zero is a regular value of Φ. Then the inclusion $\Phi^{-1}(0) \hookrightarrow Y$ induces a ring

homomorphism

$$\kappa_{\mathrm{NH}} : \mathrm{NH}_T^{*,\diamond}(Y) \to \mathrm{NH}_T^{*,\diamond}\big(\Phi^{-1}(0)\big), \tag{6.2}$$

and the latter ring is isomorphic to $H^*_{\mathrm{CR}}(Y/\!/T)$.

Moreover, under the assumption that Y^T *has only finitely many connected components,* κ_{NH} *is surjective over the rationals.*

Proof

Since Y is symplectic, its tangent bundle has a canonical T-invariant almost complex structure up to isotopy (no "stably" required). To show that $\Phi^{-1}(0)$ is stably almost complex and that κ_{NH} is a ring homomorphism, we apply Proposition 5.1, so now we establish its two requirements. Both use the exact sequence

$$0 \to T\Phi^{-1}(0) \hookrightarrow TY \twoheadrightarrow \mathfrak{t}^* \to 0,$$

which in turn depends on zero being a regular value.

For the first requirement, we can use the exact sequence to trivialize the normal bundle to $\Phi^{-1}(0)$ canonically up to isotopy. For the second, let $g \in T$, and let $y \in \Phi^{-1}(0) \cap Y^g$. The component $F \subseteq Y^g$ containing y is a Hamiltonian T-manifold with moment map $\Phi \circ i_F$, where i_F is the inclusion $F \to Y$. Since $y \in \Phi^{-1}(0)$, its $\mathfrak{t}$-stabilizer is trivial, and therefore the differential $T(\Phi \circ i_F) : T_y F \to \mathfrak{t}^*$ is onto. By the exact sequence above, this ontoness tells us that $T_y F$ is transverse to $T_y\Phi^{-1}(0)$ inside $T_y Y$. Now, apply Proposition 5.1.

Since Φ was assumed proper, $\Phi^{-1}(0)$ is compact. The isomorphism of $\mathrm{NH}_T^{*,\diamond}(\Phi^{-1}(0))$ with $H^*_{\mathrm{CR}}(Y/\!/T)$ then follows immediately from Theorem 4.1.

For each $g \in T$, we have $(Y^g)^T = Y^T$, so $(Y^g)^T$ has only finitely many connected components. Hence we can apply ordinary Kirwan surjectivity (Theorem 6.1) to each map

$$H_T^*(Y^g; \mathbb{Q}) \to H_T^*\big(\Phi^{-1}(0)^g; \mathbb{Q}\big).$$

Summing these together, we find that the map κ_{NH} is surjective over the rationals. □

Kirwan's result gives an implicit description of the kernel of κ. Tolman and Weitsman [TW] give an explicit description of the kernel, which is useful to compute the kernel for reductions of coadjoint orbits.

THEOREM 6.5 (Tolman and Weitsman [TW, Theorem 1])

Let Y *be a compact Hamiltonian* T*-space. Let* $(Y^T)_{\mathrm{cc}}$ *be the set of connected components of the fixed point set* Y^T*. Choose any* $\xi \in \mathfrak{t}$*, and let*

$$K_\xi = \big\{\alpha \in H_T^*(Y) : \alpha|_F = 0 \textit{ for all } F \in (Y^T)_{\mathrm{cc}} \textit{ such that } \langle \Phi(F), \xi\rangle \geq 0\big\}.$$

The kernel of the Kirwan map κ in equation (6.1) is given by the ideal

$$\ker \kappa = \Big\langle \bigcup_{\xi \in \mathfrak{t}} K_\xi \Big\rangle.$$

The methods introduced in [TW] allow us to generalize to the case where Y is not compact but is a proper Hamiltonian T-space; that is, its moment map has a component that is bounded from below. This applies, for example, in the case where $Y = \mathbb{C}^n$ with a proper moment map. We rephrase the theorem in this light.

We begin by asserting the existence of certain natural cohomology classes. Let $\Phi^\xi = \langle \Phi, \xi \rangle$ be a component of the moment map. We follow [GHJ] and let the *extended stable set* of F be the set of $x \in Y$ so that there is a sequence of critical sets $C_1, \dots, C_m \supseteq F$ such that x converges to an element of C_1 under the flow of $-\mathrm{grad}(f)$ (defined using a compatible Riemannian metric), and there exist points in the negative normal bundle of C_j which converge to C_{j+1} under $-\mathrm{grad}(f)$. We say that α_F is a *Morse-Thom class* associated to F if α_F is homogeneous and satisfies

(1) $\alpha_F|_{F'} = 0$ for any fixed point component $F' \in (Y^T)_{\mathrm{cc}}$ not contained in the extended stable set (with respect to Φ^ξ) of F (and, in particular, $\alpha_F|_{F'} = 0$ if $\Phi^\xi(F') < \Phi^\xi(F)$); and

(2) $\alpha_F|_F = e_T(\nu^- F)$, the equivariant Euler class of the negative normal bundle (with respect to Φ^ξ) to F.

Morse-Thom classes are not necessarily unique; there may be more than one associated to a particular F.

THEOREM 6.6

Let Y be a proper Hamiltonian T-space. Let $\kappa : H_T^*(Y) \to H^*(Y//T)$ be the Kirwan map of equation (6.1), where reduction is done at the regular value zero. The kernel of κ is generated by

$$\bigcup_{\xi \in \mathfrak{t}} \big\{ \alpha_F \in H_T^*(Y) \text{ such that } F \in (Y^T)_{\mathrm{cc}} \text{ and } \Phi^\xi(F) > 0 \big\},$$

where α_F is any Morse-Thom class associated to F.

We can use Theorems 6.5 and 6.6 to find the kernel of the surjection (6.2).

COROLLARY 6.7

Let Y be a proper Hamiltonian T-space, and let $\kappa_{\mathrm{NH}} : \mathrm{NH}_T^{*,\diamond}(Y) \to H_{\mathrm{CR}}(Y//T)$ be the natural surjection. Then

$$\ker \kappa_{\mathrm{NH}} = \bigoplus_{g \in T} \ker \kappa_g,$$

where $\ker \kappa_g$ is generated by those $\alpha_F \in H_T^*(Y^g)$ described in Theorem 6.6.

One immediate observation from Corollary 6.7 is that, for most $g \in T$, the entire piece $\mathrm{NH}_T^{*,g}(Y) = H_T^*(Y^g)$ is in the kernel. This follows from the fact that, for generic g, $Y^g = Y^T$ and does not intersect $\Phi^{-1}(0)$. Indeed, the only values of $g \in T$ such that $\mathrm{NH}_T^{*,g}(Y)$ is not contained in $\ker \kappa_{\mathrm{NH}}$ are those such that Y^g has an effective T-action. In other words, they are finite stabilizers. We find a smaller ring that surjects onto $H_{\mathrm{CR}}^*(Y//T)$ by excluding those $\mathrm{NH}_T^{*,g}(Y)$ in $\mathrm{NH}_T^{*,\diamond}(Y)$ so that g is not a finite stabilizer.

Definition 6.8
An element $g \in T$ is a *finite stabilizer* if there exists a point $y \in Y$ with $\mathrm{Stab}(y)$ finite and $g \in \mathrm{Stab}(y)$. We let Γ denote the group generated by all finite stabilizers of Y in T. We assume that Γ is finite, which is automatic if Y is of finite type.

Remark 6.9
If $g \in \Gamma$, then $a_{\lambda_j}(g)$, $j = 1, \ldots, n$, are rational numbers. In other words, the grading restricted to $\mathrm{NH}_T^{*,\Gamma}(Y)$ is rational. This accounts for the rational grading on the Chen-Ruan cohomology of the quotient space (see [FG], [R]).

LEMMA 6.10
An element $g \in T$ is a finite stabilizer on Y if and only if there exists $p \in Y^T$ such that the weights $\lambda \in \widehat{T}$ of T_pY with logweight $a_\lambda(g) = 0$ linearly span the weight lattice $\widehat{T}$ (over $\mathbb{Q}$).

Proof
First, we note that an element g is a finite stabilizer if and only if Y^g contains a component on which the generic T-stabilizer is finite. Equivalently, the $\mathfrak{t}$-stabilizer should be trivial.

Let F be a component of Y^g on which the stabilizer has minimum dimension, and let $p \in F^T$. Then the generic stabilizer on F is the same as the generic stabilizer on T_pF, which is the intersection of the kernels of the weights λ on T_pF. For this intersection to be zero, then dually the weights λ should span $\widehat{T}$. □

Definition 6.11
The Γ-*subring* $\mathrm{NH}_T^{*,\Gamma}(Y)$ of $\mathrm{NH}_T^{*,\diamond}(Y)$ is given as an $H_T^*(pt)$-module by

$$\mathrm{NH}_T^{*,\Gamma}(Y) := \bigoplus_{g\in\Gamma} \mathrm{NH}_T^{*,g}(Y).$$

It follows from Remark 3.5 that $(\mathrm{NH}_T^{*,\Gamma}(Y), \smile)$ is a subring of $\mathrm{NH}_T^{*,\diamond}(Y)$.

The following corollary immediately follows.

COROLLARY 6.12
Let Y be a proper Hamiltonian T-space. Suppose that Y^T has only finitely many connected components. By abuse of notation, we write κ_{NH} for $\kappa_{\mathrm{NH}}|_{\mathrm{NH}_T^{*,\Gamma}(Y)}$. Then

$$\kappa_{\mathrm{NH}} \;:\; \mathrm{NH}_T^{*,\Gamma}(Y;\mathbb{Q}) \longrightarrow\!\!\!\!\rightarrow \mathrm{NH}_T^{*,\diamond}\big(\Phi^{-1}(0);\mathbb{Q}\big) \cong H^*_{\mathrm{CR}}(Y/\!/T;\mathbb{Q}), \tag{6.3}$$

where the reduction is taken at any regular value of the moment map. As before,

$$\ker \kappa_{\mathrm{NH}} = \bigoplus_{g\in\Gamma} \ker \kappa_g,$$

where $\ker \kappa_g$ *is generated by those $\alpha_F \in H_T^*(Y^g)$ described in Theorem 6.6.*

Thus $H^*_{\mathrm{CR}}(Y/\!/T)$ may be computed by finding the (finitely generated) ring $\mathrm{NH}_T^{*,\Gamma}(Y)$ and quotienting by the (finitely generated) $\ker \kappa_{\mathrm{NH}}$. We show several such computations in Sections 8 and 9.

7. A graphical view for the product on Hamiltonian T-spaces and the inertial surjection (1.3)

In this section, we assume that Y has isolated fixed point set Y^T. The most commonly studied examples are toric varieties and flag manifolds, but we even find interest in the case of Y, a vector space with T acting linearly. Very shortly, we also require Y to be a proper Hamiltonian T-space and, in particular, robustly equivariantly injective.

Using the standard ring structure on $H_T^*(Y^T)\otimes_{\mathbb{Z}}\mathbb{Z}[T]$ (i.e., *not* our multiplication $\star$), the natural restriction $\mathrm{NH}_T^{*,\diamond}(Y) \longrightarrow H_T^*(Y^T)\otimes_{\mathbb{Z}}\mathbb{Z}[T]$ obtained by restricting to the fixed point set on each piece is not usually a ring homomorphism. To make it one, we had to invent the $\star$-product on $H_T^*(Y^T)\otimes_{\mathbb{Z}}\mathbb{Z}[T]$ which twisted the multiplication using logweights.

In this section, we work the logweights into the *homomorphism* rather than the multiplication on the target, giving yet another description of the multiplication. To do so, though, we have to enlarge our base ring.

7.1. The base ring $\mathscr{H}_T$ and a new restriction map

Recall that $H_T^*(pt)$ is naturally isomorphic (over $\mathbb{Z}$) to the symmetric algebra on its degree 2 part, the weight lattice $\widehat{T}$ of the torus T. Define the commutative $H_T^*(pt)$-algebra $\mathscr{H}_T$ by

$$\mathscr{H}_T := \mathbb{Z}[\{w^r : w\in\widehat{T}, r\in\mathbb{R}_{>0}\}]/\langle w^1+v^1=(w+v)^1,\ (w^r)(w^s)=w^{r+s}\rangle,$$

in which we have included all positive real powers of our generators $H_T^2(pt)$. There is an evident inclusion of $H_T^*(pt)$ into $\mathscr{H}_T$ induced from $w\mapsto w^1$, and there is a grading on $\mathscr{H}_T$, where $\deg w^r = 2r$. It is worthy of note that including real powers into this ring has not rationalized it; in particular, $\mathscr{H}_T^0$ is still just $\mathbb{Z}$.

For $\alpha \in \mathrm{NH}^{*,g}(Y)$ (i.e., of pure degree in the second component) and $p \in Y^T$, we define the "restriction" map

$$\mathrm{res}(\alpha)|_p := \alpha|_p \prod_{\lambda \in \widehat{T}} \lambda^{\dim I_\lambda \; a_\lambda(g)} \in \mathscr{H}_T, \tag{7.1}$$

where I_λ is the λ weight space of T_pY. By summing these maps together, each tensored with $g \in \mathbb{Z}[T]$ to record the T grading, we get a map

$$\mathrm{res} : \mathrm{NH}_T^{*,\diamond}(Y) \to \Big(\bigoplus_{p \in Y^T} \mathscr{H}_T \Big) \otimes \mathbb{Z}[T],$$

which takes the place of i_{NH}^* from Section 2.

THEOREM 7.1

Let $a, b \in \mathrm{NH}_T^{*,\diamond}(Y)$, and let Y have isolated fixed points. Then $\mathrm{res} : \mathrm{NH}_T^{*,\diamond}(Y) \to \big(\bigoplus_{p \in Y^T} \mathscr{H}_T \big) \otimes \mathbb{Z}[T]$ *is a graded ring homomorphism taking $\smile$ to the ordinary product.*

Proof

It is enough to check for $a \in \mathrm{NH}_T^{*,g_1}(Y)$, $b \in \mathrm{NH}_T^{*,g_2}(Y)$. Let $p \in Y^T$. Then

$$\begin{aligned}
\mathrm{res}(a \smile b)|_p &= (a \smile b)|_p \prod_\lambda \lambda^{\dim I_\lambda \; a_\lambda(g_1 g_2)} \\
&= a|_p b|_p \prod_\lambda \lambda^{\dim I_\lambda (a_\lambda^p(g_1) + a_\lambda^p(g_2) - a_\lambda^p(g_1 g_2))} \prod_\lambda \lambda^{\dim I_\lambda \; a_\lambda(g_1 g_2)} \\
&= a|_p b|_p \prod_\lambda \lambda^{\dim I_\lambda (a_\lambda^p(g_1) + a_\lambda^p(g_2))} \\
&= \Big(a|_p \prod_\lambda \lambda^{\dim I_\lambda a_\lambda^p(g_1)} \Big) \Big(b|_p \prod_\lambda \lambda^{\dim I_\lambda a_\lambda^p(g_2)} \Big) \\
&= (\mathrm{res}\; a)|_p (\mathrm{res}\; b)|_p .
\end{aligned}$$

To get from the first to the second line, we used Theorem 3.6.

To check the grading, we assume that a is of pure degree in $\mathrm{NH}_T^{*,\diamond}(Y)$, for example, if $a \in \mathrm{NH}_T^{*,g}(Y) = \bigoplus_{F \subseteq Y^g} H_T^*(F)$ is actually a homogeneous element of $H_T^*(F)$ for some component F of Y^g. Then, for p any element of F^T,

$$\begin{aligned}
\deg \mathrm{res}\; a = \deg_{\mathscr{H}_T} (\mathrm{res}\; a)|_p &= \deg_{\mathscr{H}_T} \Big(\alpha|_p \prod_{\lambda \in \widehat{T}} \lambda^{\dim I_\lambda \; a_\lambda(g)} \Big) \\
&= (\deg \alpha_p) + \sum_\lambda 2 \dim I_\lambda \; a_\lambda(g),
\end{aligned}$$

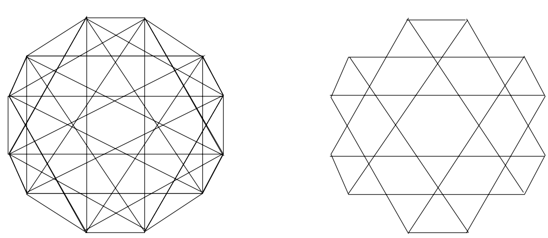
which is exactly the age-shifted definition we gave for the grading on the $H_T^*(F)$-component of $\mathrm{NH}_T^{*,g}(Y)$. □

Remark 7.2
The kernel of res is the same as the kernel of i_{NH}^*. In particular, if Y is robustly equivariantly injective, we can use Theorem 7.1 to compute the $\smile$-product.

7.2. A pictorial description of the product
In this section, we assume that Y is a proper Hamiltonian T-space, as this allows us to read off the finite stabilizers from information that is often recorded with the *moment polyhedron* $\Phi(Y)$. We recall the basic facts that we need from the geometry of moment maps, as can be found in, for example, [GS1]. The pictorial description extends that used in ordinary equivariant cohomology, as detailed in, for example, [HHH].

The image $\Phi(Y) \subseteq \mathfrak{t}^*$ of Y under the moment map Φ is a convex polyhedron (possibly unbounded), and when Y is compact, it is the convex hull of the finite set $\Phi(Y^T)$, then called the *moment polytope*. For $p \in Y^T$ and λ a weight of T_pY, the component F of $Y^{\ker\lambda}$ containing the point p is itself a proper Hamiltonian T-space. Its moment polyhedron $\Phi(F)$ is an *interval* inside $\Phi(Y)$ with one end at $\Phi(p)$, continuing in the direction λ. (To think of $\lambda \in \widehat{T}$ as a vector in $\mathfrak{t}^*$, we are using the natural embedding $\widehat{T} \to \mathfrak{t}^*$.) When we draw moment polyhedra, we always superimpose on them these intervals, which include the edges of the polyhedron $\Phi(Y)$. Therefore, from the picture alone, one can almost determine the weights of the T-action on T_pY, but only up to positive scaling and multiplicity. *We assume that we know the actual weights.*

Given $g \in T$, the moment map image $\Phi(Y^g)$ may not be convex since Y^g is not necessarily connected. The moment map image for a generic coadjoint orbit $\mathcal{O}$ of G_2 under the action of its maximal torus is shown in Figure 7.1(a). Let T be a maximal torus of G_2, and let $g \in T$ be an element of order 3 which fixes two copies of a generic coadjoint orbit of SU(3) inside. The image of $\mathcal{O}^g$ can be seen in Figure 7.1(b).



(a) (b)

Figure 7.1. (a) The moment map image of a generic coadjoint orbit of G_2; (b) the image of the fixed point set of a special order 3 element of $T \subset G_2$

There is a pictorial way to represent elements of $\mathscr{H}_T$ (and later, $\mathrm{NH}_T^{*,\diamond}(Y)$) using the fact that $\mathfrak{t}^*$ plays two roles; it is the home of the moment polytope $\Phi(Y)$ and also consists of the generators of $H_T^*(pt)$. Each monomial $w_1^{r_1} w_2^{r_2} \cdots w_k^{r_k} \in \mathscr{H}_T$ may be *drawn* by drawing the vectors $w_1, \ldots, w_k$ and labeling each w_i with the positive real number r_i. A sum of such monomials may be drawn as the formal sum of such vector drawings, one for each monomial.

As noted previously, the res map is injective for Hamiltonian spaces. So, to draw a class $\alpha_g \in \mathrm{NH}_T^{*,g}(Y)$, we can consider its image under the res map.

Let p be a fixed point that we assume to be isolated. Note that $\mathrm{res}(\alpha_g)|_p$ may be expressed as a product of two elements of $\mathscr{H}_T$:

- $\prod_{\lambda \in \widehat{T}} \lambda^{\dim I_\lambda(a_\lambda(g))}$, which depends on g but not on α;
- $\alpha_g|_p$.

Suppose that we know $(\alpha_g)|_p$ explicitly and that it can be written as a monomial $c \cdot w_1^{r_1} \cdots w_k^{r_k}$ with $r_i \in \mathbb{Z}$. We draw $w_1^{r_1} \cdots w_k^{r_k}$ at $\Phi(p)$, and we draw any coefficient c as a number at $\Phi(p)$. We draw the product $\prod_{\lambda \in \widehat{T}} \lambda^{\dim I_\lambda \; a_\lambda(g)}$ near the moment image $\Phi(X^g)$, close to $\Phi(p)$. This has the advantage of separating these two different pieces of the computation of $\mathrm{res}(\alpha_g)$. If $(\alpha_g)|_p$ is a sum of such monomials, then $\mathrm{res}(\alpha_g)|_p$ is a sum of these labeled-vector drawings, one for each monomial. For later purposes, we caption each labeled-vector drawing with the element g as well. If $\alpha_g|_p$ is not given explicitly, it may be more convenient to draw the vectors in slightly different positions (see Figure 7.3).

There is a slight annoyance if two fixed points p and q have $\Phi(p) = \Phi(q)$ in that we have to move the picture of one of them. (This is not a mathematical objection, just a practical one.)

We may now draw $\mathrm{res}(\alpha_g)$ by associating a sum of diagrams to each $p \in Y^T$. In particular, if $\mathrm{res}(\alpha_g)|_p$ is a monomial for every p, then one diagram suffices to represent α_g. In Figure 7.2, we draw the picture corresponding to $i_{\mathrm{NH}}^*(\alpha_g) = \alpha_g|_{Y^T}$, and we draw that corresponding to $\mathrm{res}(\alpha_g)$.

While each α_g may be written as a class in $H_T^*(Y^g)$, $\mathrm{res}(\alpha_g)$ may not be drawn on the moment map for Y^g (using only the weights of T on T_pY^g for each p). At any fixed p, the vectors λ occurring in the term $\prod_{\lambda \in \widehat{T}} \lambda^{\dim I_\lambda \; a_\lambda(g)}$ of equation (7.1) point out of Y^g; they are by definition those λ occurring in T_pY whose logweights $a_\lambda(g)$ are *not* zero.

Multiplication of two classes $\alpha_g \in \mathrm{NH}_T^{*,g}(Y)$ and $\beta_h \in \mathrm{NH}_T^{*,h}(Y)$ is easy in this pictorial calculus. The product of classes is performed pointwise and involves only the product structure on $\mathscr{H}_T$ with no additional factors such as those introduced by the $\star$-product since they have been worked into res. The $\mathbb{Z}[T]$-factor is only there to remember that $\alpha_g\beta_h$ lives in $\mathrm{NH}_T^{*,gh}(Y)$.

By distributivity, it is enough to treat the case where each $\alpha_g|_p$ or $\beta_h|_p$ is a monomial. The product of a diagram labeled by g and one labeled by h is labeled by

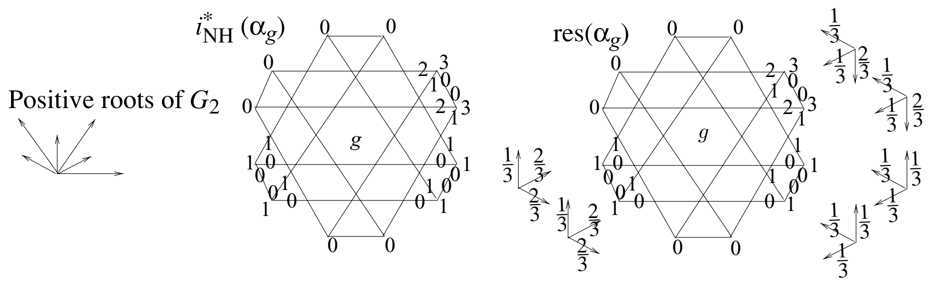


Figure 7.2. The restrictions $i^*_{\mathrm{NH}}(\alpha_g)$ and res(α_g), drawn on the picture $\mathcal{O}^g$, the fixed point set under g of a generic coadjoint orbit of G_2, where g is an element of order 3 fixing two coadjoint orbits of SU(3)

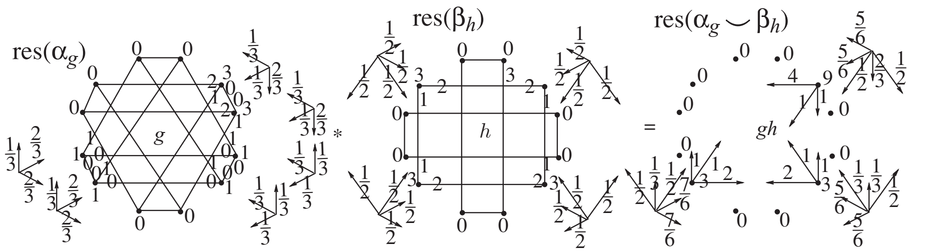


Figure 7.3. The product of two diagrams

gh. The label (exponent) on a vector λ at p in the product is the sum of the labels at p in the g-diagram and the h-diagram.

For example, let g be an order 3 element in the maximal torus of G_2 fixing two copies of SU(3)$/T$ in G_2/T, and let h be an order 2 element fixing three copies of SO(4)$/T$. Two elements α_g and β_h and their product are described by the diagrams in Figure 7.3.

Note that the final picture is obtained by adding the labels of each of the vectors but by multiplying the coefficients at $\Phi(p)$ for each fixed p. The result is a pictorial representation of res($\alpha_g \smile \beta_h$); however, it is not separated into $(\alpha_g \smile \beta_h)|_p$ and $\prod \lambda^{\dim I_\lambda a_\lambda(gh)}$. The monomial drawn at $\Phi(p)$ is $\alpha_g|_p \beta_h|_p$, and the vectors near $\Phi(p)$ represent $\prod \lambda^{\dim I_\lambda (a_\lambda(g)+a_\lambda(h))}$. The products of these pieces are the same by Theorem 7.1.

7.3. Finding the finite stabilizers

It is clear at this point that calculating $\mathrm{NH}^{*,\Gamma}_T(Y)$ instead of $\mathrm{NH}^{*,\diamond}_T(Y)$ has an appeal: there are finitely many labeled diagrams such that all elements of $\mathrm{NH}^{*,\Gamma}_T(Y)$ may be expressed as $H^*_T(pt)$-linear combinations of these diagrams. And, as we noted

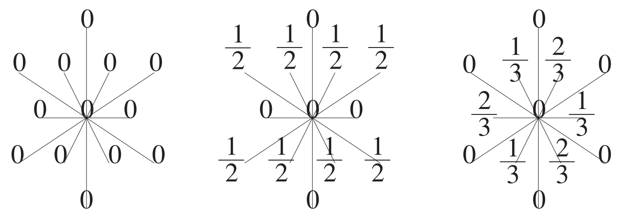


Figure 7.4. Three finite stabilizers of the action on a coadjoint orbit of G_2 pictured by their logweights on the root system

in Corollary 6.12, the ring $\mathrm{NH}_T^{*,\Gamma}(Y)$ is large enough to surject onto the Chen-Ruan cohomology of the reduced space $Y/\!/T$.

To determine the finite stabilizers, we first need a way to picture an element g of T. By Pontrjagin duality, we see that $T \cong \mathrm{Hom}(\widehat{T}, U(1))$, so we can reconstruct g from the function labeling each point $\lambda \in \widehat{T}$ by the logweight $a_\lambda(g)$. We can thus picture g as a labeling of a generating set of $\widehat{T}$ by elements of $[0, 1)$ and require that the logweights come from a homomorphism.

Lemma 6.10 said that an element g is a finite stabilizer if there exists $p \in X^T$ such that the set

$$\{\lambda \text{ is a weight of } T_pX \text{ and annihilates } g\}$$

is big enough to $\mathbb{Q}$-span $\mathfrak{t}^*$. If we assume that T acts faithfully on X, then the union over $p \in X^T$ of the weights at T_pX spans $\widehat{T}$.

We illustrate this technology to find the finite stabilizers in the example of X, a coadjoint orbit of G_2, where the union of the weights at all fixed points is exactly the root system of G_2. There are three finite stabilizers, up to rotation and reflection (the action of the Weyl group of G_2), pictured in Figure 7.4.

7.4. Finding the kernel of equation (1.3) using pictures

Before proceeding to a first, completely worked example, we illustrate the use of pictures in describing the kernel of the Kirwan map

$$\kappa_g : H_T^*(Y^g) \longrightarrow H_T^*(Y^g/\!/T)$$

and the analogous kernel of the inertial Kirwan map

$$\kappa_{\mathrm{NH}} : \mathrm{NH}_T^{*,\diamond}(Y) \longrightarrow H_{\mathrm{CR}}(Y/\!/T).$$

Recall that Corollary 6.7 states that κ_{NH} is generated by elements in the kernel of κ_g for any $g \in T$.

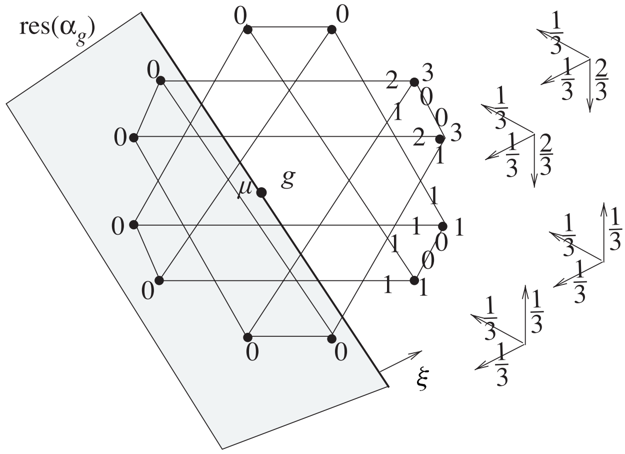


Figure 7.5. A class in the kernel of map from inertial cohomology to Chen-Ruan cohomology of the reduction

By the Tolman-Weitsman theorem (Theorems 6.5, 6.6), the kernel of κ_g is generated by classes $\alpha \in H_T^*(Y^g)$ which satisfy the property that

$$\text{there exists } \xi \in \mathfrak{t} \text{ such that } \alpha|_p = 0 \text{ for all } p \text{ with } \langle \xi, \Phi(p) \rangle \geq 0.$$

Then the kernel of κ_{NH} is generated by classes $\alpha \in \mathrm{NH}_T^{*,\diamond}(Y)$, so that $\alpha \in \mathrm{NH}_T^{*,g}(Y)$ and satisfies this property for some g.

In Figure 7.5, we show an example of an element in the kernel of the map

$$\mathrm{NH}_T^{*,\diamond}(G_2/T) \longrightarrow H_{\mathrm{CR}}(X),$$

where X is the orbifold obtained by symplectic reduction of this coadjoint orbit by T at zero. For those familiar with equivariant cohomology of coadjoint orbits (or, more generally, of GKM-spaces), one may notice that the labeling on $\Phi(Y^g)$ is compatible in some sense: the part drawn on $\Phi(Y^g)$ is the class restricted to $(Y^g)^T$ in equivariant cohomology.

To obtain the kernel of κ_{NH}, one takes every class in $\mathrm{NH}_T^{*,g}(Y)$ which has this property for some $\xi \in \mathfrak{t}$ and then does this for every picture, as g varies in T.

7.5. A toric example: $\mathbb{C}_1 \oplus \mathbb{C}_1 \oplus \mathbb{C}_3$

Let $T = S^1$, acting on $\mathbb{C}^3$ with weights 1, 1, and 3. We call the three weight lines $\mathbb{C}_1, \mathbb{C}_1', \mathbb{C}_3$. Then the finite stabilizers are $g_s = \exp(2\pi i s)$, $s = 0, 1/3, 2/3$. Encoded as functions on the weight lattice $\mathbb{Z}$, they are

$$\dots, 0, 0, \mathbf{0}, 0, 0, \dots,$$

$$\dots, \frac{1}{3}, \frac{2}{3}, \mathbf{0}, \frac{1}{3}, \frac{2}{3}, \dots,$$

$$\dots, \frac{2}{3}, \frac{1}{3}, \mathbf{0}, \frac{2}{3}, \frac{1}{3}, \dots.$$

In this case, they not only generate but are already equal to the subgroup $\Gamma \leq T$. The fixed point sets for these group elements $g_0, g_{1/3}, g_{2/3}$ are

$$\mathbb{C}_1 \oplus \mathbb{C}_1 \oplus \mathbb{C}_3, \qquad \mathbb{C}_3, \qquad \mathbb{C}_3,$$

respectively, so each $\mathrm{NH}_T^{*,g}(\mathbb{C}^3)$ is free of rank 1 over $H_T^*(pt) = \mathbb{Z}[u]$.

We calculate in detail the res map applied to the generator of $\mathrm{NH}^{*,1/3}(\mathbb{C}^3)$ at the one fixed point, $\{0\}$. This is a product over $\mathbb{C}_1, \mathbb{C}_1', \mathbb{C}_3$ of u raised to the logweight power of $g_{1/3}$, respectively, $1/3$, $1/3$, 0. Then we tensor with $g_{1/3} \in T$ to keep track of the T grading. The result is $u^{2/3} \otimes g_{1/3}$.

In all, the three generators have $\mathrm{res}|_{\{0\}}$ of

$$1 \otimes g_0, \qquad u^{2/3} \otimes g_{1/3}, \qquad u^{4/3} \otimes g_{2/3}.$$

If we call these $1, a, b$, then $a^2 = b$, $a^3 = u^2$. So

$$\mathrm{NH}^{*,\Gamma}(\mathbb{C}^3) = \frac{\mathbb{Z}[u, a, b]}{\langle a^2 - b, a^3 - u^2 \rangle} = \frac{\mathbb{Z}[u, a]}{\langle a^3 - u^2 \rangle},$$

where the bidegree of a is $(4/3, g_{1/3})$. Recall that we get an extra factor of $2 = \dim_{\mathbb{R}} \mathbb{C}$ through working with cohomology rather than Chow rings.

It is left to calculate the Chen-Ruan cohomology of the symplectic quotient at a regular value. Let us reduce at 1, and let X be the orbifold quotient; that is,

$$X = \left\{ |z_1|^2 + |z_2|^2 + 3|z_3|^2 = 1 \right\} / S^1.$$

According to the noncompact version of the Tolman-Weitsman theorem (Theorem 6.6), the kernel is generated by classes $\alpha \in \mathrm{NH}_T^{*,g}(\mathbb{C}^3)$ for each $g \in T$ whose restriction to $\{0\}$ is a multiple of the Euler class of the (negative) normal bundle to $\{0\}$. As we discuss in Section 9.2, these classes also generate the kernel when the inertial cohomology is taken with $\mathbb{Z}$-coefficients. As a module over $H_T^*(pt)$, we have

$$\mathrm{NH}_T^{*,\Gamma}(\mathbb{C}^3) = H_T^*(\mathbb{C}_1 \oplus \mathbb{C}_1 \oplus \mathbb{C}_3) \oplus H_T^*(\mathbb{C}_3) \oplus H_T^*(\mathbb{C}_3).$$

The equivariant Euler class of $\{0\}$ in the first piece is $u \cdot u \cdot 3u = 3u^3$; in the second piece, the equivariant Euler class of $\{0\}$ is $3ua$; and for the third piece, it is $3ub = 3ua^2$. We thus obtain

$$H^*_{\mathrm{CR}}(X) = \frac{\mathbb{Z}[u^{(2)}, a^{(4/3)}]}{\langle a^3 - u^2, 3u^3, 3ua\rangle},$$

where the superscripts indicate the degree.

If we drop the generator a and rationalize, we get the ordinary cohomology $H^*(X;\mathbb{Q}) = \mathbb{Q}[u]/\langle u^3\rangle$ of the coarse moduli space.

8. Flag manifolds and weight varieties

In this section, we study the example of $Y = K/T$, called a *generalized flag manifold*, where K is a compact, connected Lie group and T is a maximal torus thereof.

This example is already well handled by the techniques of Section 7, as it is Hamiltonian with isolated fixed points, and all the fixed points map to different places under the moment map. The main result of this section is then an efficient calculation of the finite stabilizers. One interesting corollary is that the obstruction bundles are all trivial if and only if K is a classical group.

The standard notation that we need from Lie theory is the normalizer of the torus $N(T)$, the Weyl group $W := N(T)/T$, and the centralizer $C_K(k)$ of an element $k \in K$. This space K/T has a left action of K (and hence of T) and a right action of W. It has a family of symplectic structures, one for each generic orbit $K \cdot \lambda$ on the dual $\mathfrak{k}^*$ of the Lie algebra of K. The moment map Φ is the projection $\mathfrak{k}^* \to \mathfrak{t}^*$ dual to the inclusion of Lie algebras. Using the Killing form, we can and do regard the base point λ as an element of $\mathfrak{t}^* \hookrightarrow \mathfrak{k}^*$.

A *weight variety* (see [Kn], [G2]) is the symplectic quotient of a coadjoint orbit $K \cdot \lambda$ by the maximal torus $T \leq K$. These turn out to be smooth for reductions at regular values for $K = \mathrm{SU}(n)$, but they are orbifolds for other K, as we explain after Proposition 8.3.

We need a few standard facts about such K:

- every element of K is conjugate to some element of T;
- if two elements of T are K-conjugate, they are already conjugate by $N(T)$;
- the center $Z(K)$ is contained in T; and
- any two maximal tori in K are conjugate.

The group K is *semisimple* if its center $Z(K)$ is finite or, equivalently, if the center of its Lie algebra is trivial.

LEMMA 8.1

Let K be a compact, connected Lie group, and let T be a maximal torus so that T contains the center $Z(K)$. Then the generic stabilizer of T acting on K/T is $Z(K)$.

In particular, unless K is semisimple, there are no finite stabilizers at all.

Proof
Since $Z(K) \leq T$, for all $z \in Z(K)$, $g \in K$ we have

$$zgT = gzT = gT.$$

Conversely, let $s \in G$ stabilize every point of K/T so that for all $k \in K$, $skT = kT$, and hence $s \in kTk^{-1}$, so s commutes with kTk^{-1}. But the union over $k \in K$ of the tori kTk^{-1} is all of K since every element can be conjugated into T. Hence s commutes with all of K. □

8.1. The finite stabilizers in T on K/T
We are now ready to determine, following [Kn], which T-stabilizers occur on K/T and their fixed points.

LEMMA 8.2
- *Let $k \in K$, and let kT be the corresponding point in K/T. Then kT is stabilized by $t \in T$ if and only if $k \in C_K(t)N(T)$.*
- *Let $C_K(t)^0$ denote the identity component of $C_K(t)$, and let W_t denote the Weyl group of $C_K(t)^0$ (with respect to the same maximal torus T). Then $C_K(t)N(T) = C_K(t)^0 N(T)$. Each component of $C_K(t)N(T)/T$ is isomorphic to the smaller flag manifold $C_K(t)^0/T$, and the components are indexed by the cosets $W_t \backslash W$.*
- *An element $t \in T$ occurs as a finite stabilizer if and only if the identity component $C_K(t)^0$ of $C_K(t)$ is semisimple and necessarily of the same rank as K.*

Proof
To start off the first claim,

$$tkT = kT \Longleftrightarrow k^{-1}tkT = T \Longleftrightarrow k^{-1}tk \in T.$$

Two elements of T are K-conjugate if and only if they are $N(T)$-conjugate. So the equivalences continue:

$$\begin{aligned}
&\Longleftrightarrow \exists w \in N(T), \quad k^{-1}tk = w^{-1}tw \\
&\Longleftrightarrow \exists w \in N(T), \quad wk^{-1}tkw^{-1} = t \\
&\Longleftrightarrow \exists w \in N(T), \quad kw^{-1} \in C_K(t) \\
&\Longleftrightarrow k \in C_K(t)N(T).
\end{aligned}$$

This chain of equivalences establishes the first claim.

For the second claim, let us first note that since T is commutative, $C_K(t) \geq T$, and since T is connected, $C_K(t)^0 \geq T$.

Plainly, $C_K(t)N(T) \supseteq C_K(t)^0 N(T)$, so our next task is to show that $C_K(t) \subseteq C_K(t)^0 N(T)$, which establishes $C_K(t)N(T) = C_K(t)^0 N(T)$. Let $c \in C_K(t)$. Then $cTc^{-1} \leq cC_K(t)^0 c^{-1} = C_K(t)^0$, so cTc^{-1} is another maximal torus of the compact connected group $C_K(t)^0$. Hence $\exists d \in C_K(t)^0$ such that $d(cTc^{-1})d^{-1} = T$. So $dc \in N(T)$, and $c \in d^{-1}N(T) \subseteq C_K(t)^0 N(T)$, completing this task.

The components of $C_K(t)^0 N(T)/T$ are the orbits of the connected group $C_K(t)^0$ through the discrete set $N(T)/T$. Let $\widetilde{w} \in N(T)$ lie over $w \in W$. Then $C_K(t)\widetilde{w}T/T = C_K(t)/Tw \cong C_K(t)/T$, as claimed. The T-fixed points on the component $C_K(t)^0 \widetilde{w}T/T$ are $W_t w$. Two components are equal if and only if their T-fixed points are the same, so the components are indexed by $W_t \backslash W$.

We turn to the third claim. Let Z denote the identity component of the center of $C_K(t)^0$. So Z is a connected subgroup of K commuting with the maximal torus T, and hence $Z \subseteq T$. We now claim that any point kT stabilized by t is also stabilized by Z.

By the first claim, we can factor k as $k = cw$, where $c \in C_K(T)^0$, $w \in N(T)$. Then, for any $z \in Z$,

$$zkT = zcwT = czwT = cwt'T = cwT = kT,$$

where since $z \in T$, we have that $t' = w^{-1}zw$ is also in T.

So, for t to occur as a finite stabilizer, Z must be trivial, meaning that $C_K(t)^0$ must be semisimple.

For the converse, we know from Lemma 8.1 that the generic T-stabilizer on $C_K(t)^0/T$ is just $Z(C_K(t)^0)$. This latter group is finite if and only if $C_K(t)^0$ is semisimple. □

The center $Z(K)$ supplies dull examples of elements of T which have semisimple centralizer (namely, all of K). If $K = \mathrm{SU}(n)$, then there are no other examples. For a first taste of what can happen in other Lie types, consider the diagonal matrix $t = \mathrm{diag}(-1, -1, -1, -1, +1)$ in $\mathrm{SO}(5)$, which has $C_K(t)^0 = \mathrm{SO}(4)$. This element fixes $|W_t \backslash W| = 2$ copies of $\mathrm{SO}(4)/T$ in $\mathrm{SO}(5)/T$.

A conjugacy class in K is called *special* if the centralizer of some (hence any) element of the class is semisimple. We typically use representatives $t \in T$, which we can do since T intersects every conjugacy class. Since semisimplicity is a Lie algebra phenomenon, it is enough to check that $\mathfrak{c}_{\mathfrak{k}}(t)$ is semisimple.

To analyze these special conjugacy classes, we run down the arguments from [BdS], where greater detail can be found. Recall that for K simple, the *affine Dynkin diagram* of K is formed from the simple roots and the *lowest* root, which we denote ω.

PROPOSITION 8.3 (see [BdS], [Dy])
Assume that K is semisimple so that there are special conjugacy classes and the universal cover $\widetilde{K}$ is again compact. The special conjugacy classes in K are images of those in $\widetilde{K}$, so it suffices to find those of $\widetilde{K}$.

Now, assume that K simple. The special conjugacy classes in $\widetilde{K}$ correspond $1:1$ *to the vertices of the affine Dynkin diagram of $\widetilde{K}$ (or of K). To find an element of the special class corresponding to a vertex v, find an element $t \in T$ annihilated by all the roots in the affine diagram* other than v. *These $t \in T$ exist and are special.*

The simple roots and the lowest root satisfy a unique linear dependence $\omega + \sum_\alpha c_\alpha \alpha = 0$ which we use to define the coefficients $\{c_\alpha\}$. The adjoint order in $K/Z(K)$ of a special element corresponding to a simple root α is c_α (e.g., a special element is central if and only if the corresponding c_α coefficient is 1).

These coefficients $\{c_\alpha\}$ can be found in, for example, [Hu, page 98]. They are all 1 for $K = \mathrm{SU}(n)/Z_n$, with the consequence that the identity is the only finite stabilizer, and the weight varieties are all manifolds. They are all 1 or 2 for the classical groups $\mathrm{SU}(n)$, $\mathrm{SO}(n)$, $U(n, \mathbb{H})$.

We now recall the role of the *Weyl alcove* in analyzing the conjugacy classes of $\widetilde{K}$.

Each conjugacy class in K meets T, and two elements of T are conjugate in K only if they are already conjugate by the action of W. So the space of conjugacy classes is $T/W = (\mathfrak{t}/\Lambda)/W = \mathfrak{t}/(\Lambda \rtimes W)$, where Λ is the *coweight lattice* $\ker(\exp : \mathfrak{t} \to T)$. If $K = \widetilde{K}$, then this semidirect product is again a reflection group, the affine Weyl group $\widehat{W}$; this is the reason that it is convenient to work with $\widetilde{K}$. This group is generated by the reflections in the hyperplanes $\langle \alpha, \cdot \rangle = 0$ for α simple and $\langle \omega, \cdot \rangle = -1$. If K is not simple, then there are several lowest roots, an uninteresting complication that we ignore.

The *Weyl alcove* $A \subset \mathfrak{t}$ is a fundamental region for $\widehat{W}$ defined by $\langle \alpha, \cdot \rangle \geq 0$ for α simple and $\langle \omega, \cdot \rangle \geq -1$. This and the analysis above ensure that for $K = \widetilde{K}$, the image $\exp(A) \subseteq T \leq K$ intersects each conjugacy class of K in exactly one point. Since the map $A \to \exp(A)$ is a homeomorphism, we may also sometimes refer to $\exp(A)$ as the Weyl alcove.

COROLLARY 8.4
If K is simply connected, then the finite stabilizers in the action of T on K/T are the Weyl conjugates of the vertices of the Weyl alcove $\exp(A)$. *If K is not simply connected, then the finite stabilizers are the images of the finite stabilizers from the universal cover $\widetilde{K}$.*

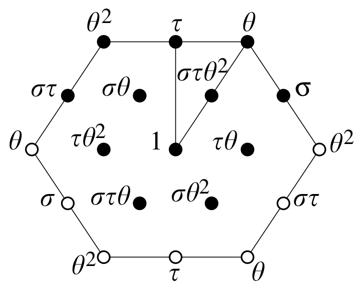


Figure 8.1. The special elements in the Tits polytope of G_2

If K is a centerless classical group, then Γ is contained in the 2-torsion subgroup of T. If K is classical but not centerless, then Γ is contained in the preimage of the 2-torsion in the torus of $K/Z(K)$.

The Weyl alcove also comes up when working examples, in that the torus T can be pictured as a quotient of the polytope $\bigcup(W \cdot A)$ made from the union of the Weyl reflections of the Weyl alcove. We call this the *Tits polytope* after its relation to the Tits cone in the corresponding affine Kac-Moody algebra and to the finite Tits building living on its surface (neither of which are relevant here). The Tits polytope tesselates the vector space $\mathfrak{t}$ under translation by the coweight lattice Λ. The exponential map $\bigcup(W \cdot A) \to T$ is onto but only $1:1$ on the interior of the polytope.

8.1.1. Example: $K = G_2$

This group is both centerless and simply connected; hence it is the unique Lie group with its Lie algebra. Its Weyl alcove is a 30°-60°-90° triangle.

We picture $\Gamma \cong Z_2 \times Z_6$ inside the Tits hexagon $\bigcup(W \cdot A) \subset \mathfrak{t}$, seen in Figure 8.1. (The Weyl alcove itself is the 30°-60°-90° triangle with vertices $\{1, \tau, \theta\}$, of orders 1, 2, 3.) This hexagon tiles the plane $\mathfrak{t}$ under translation by the coweight lattice Λ. The twelve black dots are labeled by the elements of Γ to which they exponentiate, some of which occur again as white dots. The element σ is a Weyl conjugate of τ.

The centralizers of τ and θ are SO(4) and SU(3), respectively. Hence the fixed point sets for the twelve elements of Γ are

- all of X, for $w = 1$;
- two $\mathrm{SU}(3)/T$'s for θ and θ^2;
- three $\mathrm{SO}(4)/T = (\mathbb{P}^1)^2$'s for $\tau, \sigma, \sigma\tau$; and
- six $U(2)/T = \mathbb{P}^1$'s for each of the other six elements in the interior of the hexagon.

Only the first six, which lie on *corners* of permuted Weyl alcoves, are actually finite stabilizers. The last six lie only on edges.

8.2. The inertial cohomology groups $\mathrm{NH}_T^{*,t}(K/T)$

For $t \in T$, we have already computed the fixed point set as the disjoint union $\bigcup_{w \in W_t \backslash W} C_K(t)^0 wT/T$. Therefore

$$\mathrm{NH}_T^{*,t}(K/T) = \bigoplus_{w \in W_t \backslash W} H_T^*\big(C_K(t)^0 wT/T\big).$$

The moment polytope of the component $C_K(t)^0 wT/T$ is the convex hull of the points $W_t w \cdot \lambda$.

To compute the Weyl group W_t of $C_K(t)^0$, we need only a set of reflections which generates W_t. These are the reflections in the roots perpendicular to the (possibly internal) walls of the Tits polytope passing through the point t. For there to be enough to make t a finite stabilizer, t has to lie on a vertex of a permuted Weyl alcove, as already proved in Proposition 8.3.

The cohomology $H_T^*(C_K(t)^0 wT/T)$ of one component has a basis given by the equivariant classes of Schubert varieties. For later purposes, we are also interested in the classes of permuted Schubert varieties, for which our reference is [G2].

8.3. The product structure on $\mathrm{NH}_T^{*,\diamond}(K/T)$

Let $t, s \in T$. Then, since $\mathrm{NH}_T^{*,t}(K/T)$ is the direct sum $\bigoplus_{w \in W_t \backslash W} H_T^*(C_K(t)^0 wT/T)$ and $\mathrm{NH}_T^{*,s}(K/T)$ is a similar direct sum, to understand their product it is enough to consider the product from two summands, $H_T^*(C_K(t)^0 wT/T) \times H_T^*(C_K(s)^0 vT/T)$. The definition of the $\smile$-product requires us to restrict classes from $C_K(t)^0 wT/T$ and $C_K(s)^0 vT/T$ to their intersection, multiply together and by the virtual fundamental class, and then push into $C_K(ts)^0 N(T)/T$.

LEMMA 8.5

Let $C_K(t, s)$ denote the intersection $C_K(t) \cap C_K(s)$, and let $C_K(t, s)^0$ be its identity component. Let $W_{t,s}$ denote its Weyl group.

(1) *The intersection $(C_K(t)^0 wT/T) \cap (C_K(s)^0 vT/T)$ is fixed pointwise by t and s. It is a finite union of homogeneous spaces for $C_K(t, s)^0$, namely, $\bigcup_{W_{t,s}u} C_K(t, s)^0 uT/T$, where the components are indexed by those cosets $W_{t,s}u$ contained in the intersection $W_t w \cap W_s v$.*

(2) *The obstruction bundle over $C_K(t, s)^0 uT/T$ is trivial if K is a classical group.*

Proof

Since $s \in T \leq C_K(t)^0$, we can compute $C_K(s) \cap C_K(t)^0$ as the centralizer in $C_K(t)^0$ of s. Likewise, the intersection $(C_K(t)^0 wT/T) \cap (C_K(s)^0 N(T)/T)$ can be computed as the s-fixed points on $C_K(t)^0 wT/T \cong C_K(t)^0/T$. Then apply Lemma 8.2 to the case of $C_K(t)^0$.

Now assume that K is classical, and let $g_1, g_2 \in \Gamma$. Then $g_1, g_2, (g_1 g_2)^{-1}$ each act on K/T with order 1 or 2. Hence their three logweights on any line each live in $\{0, 1/2\}$ and cannot add up to 2. So the obstruction bundle is trivial. $\square$

COROLLARY 8.6
Let K be a centerless classical group. Then the product map

$$\smile : H_T^*\big(C_K(t)^0 wT/T\big) \times H_T^*\big(C_K(s)^0 vT/T\big) \to H_T^*\big(C_K(st)^0 vT/T\big)$$

is given by

$$\alpha \smile \beta = \sum_{W_{t,s} u \subseteq W_t w \cap W_s v} (\bar{e}_3)_* \big(e_1^*(\alpha) \times e_2^*(\beta)\big),$$

where $e_1, e_2, \bar{e}_3$ are the inclusions of $C_K(s,t)^0 uT/T$ into $C_K(t)^0 uT/T$, $C_K(s)^0 uT/T$, $C_K(ts)^0 uT/T$, respectively.

The maps e_i^* and $(e_i)_*$ between the nonequivariant cohomologies of these homogeneous spaces have been studied in [P], in part for the application in [BS] to asymptotic branching rules.

8.3.1. Example: $K = \mathrm{SO}(5)$
The Weyl alcove of $\widetilde{K} = \mathrm{Spin}(5)$ is a 45°-45°-90° triangle, and the group Γ in $\widetilde{K}$ is exactly the 2-torsion in T, all of whose elements are finite stabilizers. Its quotient in SO(5) is the two-element group $\{\mathbf{1}, t := \mathrm{diag}(-1,-1,-1,-1,+1)\}$.

Hence there are only two summands in $\mathrm{NH}_T^{*,\Gamma}(\mathrm{SO}(5)/T)$. By Remark 2.4, the only difficult product is from the t-summand, squared, back to the identity summand. In this case, e_1 and e_2 are the identity, so the only map of interest is $(\bar{e}_3)_* : H_T^*\big((\mathrm{SO}(5)/T)^t\big) \to H_T^*(\mathrm{SO}(5)/T)$. This is perhaps best computed via the techniques in Section 7.

8.4. The kernel of the inertial Kirwan map
Finally, we need to compute the kernel of the map from $\mathrm{NH}_T^{*,\Gamma}(K \cdot \lambda)$ to $H_{\mathrm{CR}}^*(K \cdot \lambda //_\mu T)$. Breaking this up by $t \in \Gamma$ and then into components of $(K \cdot \lambda)^t$, this kernel is the direct sum of the kernels of each of the ordinary Kirwan maps

$$H_T^*\big(C_K(t)^0 u \cdot \lambda\big) \to H^*\big(C_K(t)^0 u \cdot \lambda //_\mu T\big).$$

This kernel is computed in [G2]; it is spanned by the classes of those permuted Schubert varieties whose images under the moment map miss μ.

9. Toric varieties

In this section, we use our results to compute the Chen-Ruan cohomology of certain toric orbifolds. We first discuss the symplecto-geometric construction of toric orbifolds as described by Lerman and Tolman [LT]. We remark on the coefficients in the toric case, and we compute an example in full detail. Finally, we relate our results to those of Borisov, Chen, and Smith [BCS].

9.1. Symplectic toric orbifolds

In [LT], Lerman and Tolman study Hamiltonian torus actions on symplectic orbifolds and define and classify symplectic toric orbifolds. They consider the case where the orbifold is *reduced*: there is no global stabilizer. These reduced toric orbifolds, then, are in one-to-one correspondence with labeled simple rational polytopes.

Definition 9.1

Let $\mathfrak{t}$ be a d-dimensional vector space with a distinguished lattice ℓ; let $\mathfrak{t}^*$ be the dual space. A convex polytope $\Delta \subset \mathfrak{t}^*$ is *rational* if

$$\Delta = \bigcap_{i=1}^{N} \left\{ \alpha \in \mathfrak{t}^* \,\middle|\, \langle \alpha, y_i \rangle \geq \eta_i \right\}$$

for some $y_i \in \ell$ and $\eta_i \in \mathbb{Q}$. A *facet* is a face of codimension 1. A d-dimensional polytope is *simple* if exactly d facets meet at every vertex. A *labeled polytope* is a convex rational simple polytope along with a positive integer labeling each facet.

To establish a one-to-one correspondence between symplectic toric orbifolds and labeled polytopes, Lerman and Tolman mimic Delzant's construction of (smooth) toric varieties as symplectic quotients. The labeled polytope Δ is uniquely described as the intersection of half-spaces,

$$\Delta = \bigcap_{i=1}^{N} \left\{ \alpha \in \mathfrak{t}^* \,\middle|\, \langle \alpha, y_i \rangle \geq \eta_i \right\},$$

where N is the number of facets, the vector $y_i \in \ell$ is the primitive inward-pointing normal vector to the ith facet, and m_i is the positive integer labeling that facet. Define the map $\varpi : \mathbb{R}^N \to \mathfrak{t}$ defined by sending the ith standard basis vector e_i to $m_i y_i$. This yields a short exact sequence,

$$0 \longrightarrow \mathfrak{k} \xrightarrow{j} \mathbb{R}^N \xrightarrow{\varpi} \mathfrak{t} \longrightarrow 0, \tag{9.1}$$

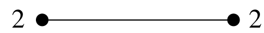


Figure 9.1. The labeled polytope corresponding to $\mathbb{C}^2//(S^1 \times \mathbb{Z}_2)$

and its dual,

$$0 \longrightarrow \mathfrak{t}^* \xrightarrow{\varpi^*} (\mathbb{R}^N)^* \xrightarrow{j^*} \mathfrak{k}^* \longrightarrow 0, \tag{9.2}$$

where $\mathfrak{k} = \ker(\varpi)$. Let $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ and $T = \mathfrak{t}/\ell$, and let K denote the kernel of the map $\mathbb{T}^N \to T$ induced by ϖ. Then the Lie algebra of K is $\mathfrak{k}$. Tolman and Lerman then prove that $Y_\Delta = \mathbb{C}^N//K$ is the unique symplectic toric orbifold with moment polytope Δ.

As noted above, this construction does not include the possibility of a global finite stabilizer. It may, however, result in a quotient by a disconnected subgroup K of $\mathbb{T}$. For example, for the polytope shown in Figure 9.1, using the above construction, we find that $K \cong S^1 \times \mathbb{Z}_2$. In order to obtain a connected group K, it is sufficient, though not necessary, to assume that the labels are all 1. When K is connected, we may use the techniques developed in Sections 2 and 6 to compute the Chen-Ruan cohomology of Y_Δ. When K is disconnected, we need an additional argument to see that the surjectivity remains true.

PROPOSITION 9.2
Let $Y_\Delta = \mathbb{C}^N//K$ be a symplectic toric orbifold. Then there is a surjection

$$H_K^*(\mathbb{C}^N;\mathbb{Q}) \longrightarrow H^*(Y_\Delta;\mathbb{Q}).$$

Proof
We can write $K = T \times G$, where T is a connected torus and G is a finite abelian group (possibly trivial). Since G is finite, we may identify

$$H_{T\times G}^*(\mathbb{C}^N;\mathbb{Q}) \cong H_T^*(\mathbb{C}^N/G;\mathbb{Q}).$$

Similarly, for a level set $\Phi^{-1}(0)$, there is an isomorphism $H_{T\times G}^*(\Phi^{-1}(0);\mathbb{Q}) \cong H_T^*(\Phi^{-1}(0)/G;\mathbb{Q})$. Now $\mathbb{C}^N/G$ is a symplectic orbifold, and T acts on $\mathbb{C}^N/G$ in a Hamiltonian fashion. Moreover, the level set for a T moment map may be identified as $\Phi^{-1}(0)/G$, the K-level set modulo G. One may extend Kirwan surjectivity to orbifolds using the techniques detailed in [LMTW] and thus conclude surjectivity of

$$H_T^*(\mathbb{C}^N/G;\mathbb{Q}) \longrightarrow H_T^*\big(\Phi^{-1}(0)/G;\mathbb{Q}\big).$$

Finally, we identify $H_T^*(\Phi^{-1}(0)/G;\mathbb{Q}) \cong H^*(Y_\Delta;\mathbb{Q})$, completing the proof. $\square$

From the labeled moment polytope, it is not difficult to determine the stabilizer group of any point. There are two contributions: a contribution that can be seen in the underlying polytope, and a contribution from the facet labels. Given a point in Y_Δ, let f be the minimal face in Δ which contains its image under the moment map, let V be the subspace containing f, and let $V_{\mathbb{Z}} = V \cap \mathbb{Z}^N$. Let E be the set of primitive vectors incident to a vertex of f which are not in the subspace V. Let π be the projection from $\mathbb{Z}^N$ to $\mathbb{Z}^N / V_{\mathbb{Z}}$. The contribution to the stabilizer coming from the polytope is the group

$$\Gamma_f^1 = (\mathbb{Z}^N / V_{\mathbb{Z}})/\mathrm{Span}_{\mathbb{Z}}\big(\pi(E)\big).$$

Now we let F_i denote the facet of Δ with label ℓ_i. The contribution to the stabilizer from the labels is

$$\Gamma_f^2 = \gcd(\ell_i \mid f \subseteq F_i),$$

and so the stabilizer to the point we are considering is

$$\Gamma_f = \Gamma_f^1 \times \Gamma_f^2. \tag{9.3}$$

9.2. A comment on coefficients

The surjectivity statement in Theorem 6.4 follows directly from Kirwan's Theorem 6.1. As a consequence, Theorem 6.4 holds over $\mathbb{Z}$ whenever Theorem 6.1 holds over $\mathbb{Z}$ for each of the orbistrata. Even for a smooth toric variety, however, it is not immediately clear (though it is true; see [HT]) that Kirwan's proof of surjectivity generalizes to integer coefficients. Here we first establish Kirwan's result over $\mathbb{Z}$ for weighted projective spaces.

PROPOSITION 9.3

Suppose that S^1 acts on $\mathbb{C}^N$ linearly with positive weights. Then $\mathbb{C}^N // S^1$ is a weighted projective space*, and the ring homomorphism*

$$\kappa \;:\; H^*_{S^1}(\mathbb{C}^N;\mathbb{Z}) \to H^*(\mathbb{C}^N // S^1;\mathbb{Z}) \tag{9.4}$$

is a surjection.

Proof

Since S^1 acts linearly on $\mathbb{C}^N$ with positive weights, the moment map is of the form

$$\Phi(z_1,\ldots,z_N) = b_1|z_1|^2 + \cdots + b_N|z_N|^2 + C,$$

where $b_i \geq 0$ are positive multiples of the positive weights of the circle action and C is a constant. Hence a level set at a regular value is homeomorphic to a sphere, and the resulting quotient S^{2N-1}/S^1 is precisely a weighted projective space. Indeed, every weighted projective space arises in this way.

Surjectivity holds over $\mathbb{Z}$ in these examples by results similar to [TW, Propositions 7.3, 7.4], where Tolman and Weitsman show that surjectivity over $\mathbb{Z}$ depends only on the integral cohomology of the circle fixed points being torsion-free. Although Tolman and Weitsman's results require the original manifold to be compact, because we have the facts that

(1) Φ is proper and bounded below, and

(2) $\mathbb{C}^N$ has only finitely many fixed point components,

we may generalize their propositions to this setting. Thus, for surjectivity to hold over $\mathbb{Z}$, it is sufficient for the integral cohomology of each fixed point component to be torsion-free. Now, in this case, the only circle fixed point is the origin in $\mathbb{C}^N$, and of course, a point has torsion-free integral cohomology. This completes the proof. □

In a preliminary version of this article, we stated a stronger (but incorrect) version of Proposition 9.3. It applied to a class of toric orbifolds with labeled polytopes of a certain combinatorial type, including, but not limited to, weighted projective spaces. To prove it, we established that the critical sets of $\|\Phi\|^2$ have torsion-free integral cohomology. This condition is in fact not sufficient to deduce surjectivity over $\mathbb{Z}$. Indeed, the manifold X in [TW, Lemma 7.1] must be *pointwise* fixed by the torus T, so we may only apply this Lemma to the T-fixed points of the original manifold; however, unless T is a circle, there are critical sets C of $\|\Phi\|^2$ which are only fixed by a subtorus K of T. As noted in Remark 6.3, to prove surjectivity over $\mathbb{Z}$, we need not only that $H^*(C;\mathbb{Z})$ be torsion-free but also that $H_K^*(C;\mathbb{Z})$ be torsion-free.

Theorem 6.1 does hold over $\mathbb{Z}$ more generally, though not for all toric orbifolds. For example, consider the product of weighted projective spaces $X = \mathbb{P}_{1,1,2} \times \mathbb{P}_{1,1,2}$. Using the Künneth theorem over $\mathbb{Z}$, it is straightforward to compute the cohomology of this space. Because $\mathbb{P}_{1,1,2}$ has 2-torsion in its cohomology, the Tor terms do not vanish, and as a result, the cohomology of the product has 2-torsion in certain odd degrees. As a result, surjectivity cannot possibly hold over $\mathbb{Z}$ because the equivariant cohomology of affine space has cohomology only in even degrees. The failure in this example is entirely due to the fact that there is a point with stabilizer $\mathbb{Z}_2 \times \mathbb{Z}_2$. For a toric orbifold whose polytope has labels all equal to one, to ensure surjectivity over $\mathbb{Z}$, it is at the very least necessary to assume that the isotropy at each point of the orbifold is a cyclic group. This is being more closely investigated by Holm and Tolman [HT].

We now use Proposition 9.3 to deduce an integral version of Theorem 6.4 for a weighted projective space $\mathbb{C}^N/\!/S^1$. Each orbistratum is itself a circle reduction of some $\mathbb{C}^k \subseteq \mathbb{C}^N$ and thus also a weighted projective space, immediately implying surjectivity over $\mathbb{Z}$ for each piece of the inertial cohomology. Hence we have the following corollary.

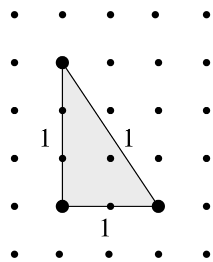


Figure 9.2. The labeled polytope corresponding to $(\mathbb{C}_{(1)} \oplus \mathbb{C}_{(2)} \oplus \mathbb{C}_{(3)})//S^1$

COROLLARY 9.4
Suppose that S^1 acts on $\mathbb{C}^N$ linearly with positive weights. Then the ring homomorphism

$$\kappa_{\mathrm{NH}} : \mathrm{NH}^{*,\diamond}_{S^1}(\mathbb{C}^N;\mathbb{Z}) \to H^*_{\mathrm{CR}}(\mathbb{C}^N//S^1;\mathbb{Z}) \tag{9.5}$$

is a surjection.

Finally, we note that whenever the surjectivity result holds over $\mathbb{Z}$ for each orbistratum, the kernel computations of Theorem 6.6 extend to inertial cohomology with $\mathbb{Z}$-coefficients. As the examples presented below are all weighted projective spaces or smooth toric varieties, surjectivity and the kernel computations do hold over $\mathbb{Z}$.

9.3. The Chen-Ruan cohomology of a weighted projective space
We now present an example to demonstrate the ease of computation of inertial cohomology and of the kernel of the surjection to Chen-Ruan cohomology of the reduction.

Example 9.5
Let Δ be the moment polytope in Figure 9.2.

In this case, if we let $Y = \mathbb{C}^3$ and follow the above construction, then $K \cong S^1$ acts on Y by

$$t \cdot (z_1, z_2, z_3) = (t \cdot z_1, t^2 \cdot z_2, t^3 \cdot z_3).$$

Then as an S^1-representation, $Y = \mathbb{C}_{(1)} \oplus \mathbb{C}_{(2)} \oplus \mathbb{C}_{(3)}$. This action is Hamiltonian with moment map

$$\Phi(z_1, z_2, z_3) = |z_1|^2 + 2|z_2|^2 + 3|z_3|^2.$$

Any positive real number is a regular value of Φ, and the symplectic reduction is a weighted projective space $Y_\Delta = Y//S^1 = \mathbb{P}^2_{1,2,3}$. Changing the regular value at which we reduce only changes the symplectic form on Y_Δ. We now compute the Chen-Ruan cohomology of the symplectic reduction by computing $\mathrm{NH}^{*,\Gamma}_{S^1}(Y)$ and computing the kernel of the surjection $\kappa : \mathrm{NH}^{*,\Gamma}_{S^1}(Y) \to H^*_{\mathrm{CR}}(Y_\Delta)$. In this case, the surjection actually holds over $\mathbb{Z}$, so we assume integer coefficients for the remainder of the example.

First, we notice that the finite stabilizers for the S^1-action are the square and cube roots of unity inside S^1. Thus the group that they generate is the set $\Gamma = \{\zeta_k = \exp(2\pi ik/6) \mid k = 0, \ldots, 5\} \cong \mathbb{Z}_6$ of sixth roots of unity. For each ζ_i, Y^{ζ_i} is contractible, so $\mathrm{NH}^{*,\zeta_i}_{S^1}(Y)$ is free of rank 1. To compute $\mathrm{NH}^{*,\Gamma}_{S^1}(Y)$, we now refer to Table 9.1.

Table 9.1. Computing the degrees of the pieces of $\mathrm{NH}^{*,\Gamma}_{S^1}(Y)$

g	ζ_0	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
Y^g	Y	$\{0\}$	$\mathbb{C}_{(3)}$	$\mathbb{C}_{(2)}$	$\mathbb{C}_{(3)}$	$\{0\}$
$a_{\mathbb{C}_{(1)}}(g)$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
$a_{\mathbb{C}_{(2)}}(g)$	0	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$
$a_{\mathbb{C}_{(3)}}(g)$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$
2 age(g)	0	2	2	2	2	4
generator of $\mathrm{NH}^{*,g}_{S^1}(Y)$	1	α	β	γ	δ	η

Thus $\mathrm{NH}^{*,\Gamma}_{S^1}(Y)$ is generated as a free module over $H^*_{S^1}(pt)$ by the elements $1, \alpha, \beta, \gamma, \delta$, and η. Let u denote the degree 2 generator of $H^*_{S^1}(pt)$. We determine the product structure by computing all pairwise products of these generators.

Remark 9.6
By Corollary 3.7, we may abuse notation and write $\star$ for the product on $\mathrm{NH}^{*,\Gamma}_{S^1}(Y)$.

We demonstrate the calculation of $\eta \star \eta$ using (2.1) and then give the multiplication table. We know that $\eta \star \eta \in \mathrm{NH}^{*,\zeta_4}_{S^1}(Y)$, so we compute what multiple of δ it is. Since η is supported on the $(g = \zeta_5)$-piece of Y, we compute the logweights of ζ_5 and ζ_4 on νY^g, the normal bundle to the fixed point set $Y^g = \{0\}$. Here $\nu Y^g = \mathbb{C}_{(1)} \oplus \mathbb{C}_{(2)} \oplus \mathbb{C}_{(3)}$. We find that

$$a^{\{0\}}_{(1)}(\xi_5) + a^{\{0\}}_{(1)}(\xi_5) - a^{\{0\}}_{(1)}(\xi_4) = \frac{5}{6} + \frac{5}{6} - \frac{2}{3} = 1,$$

$$a^{\{0\}}_{(2)}(\xi_5) + a^{\{0\}}_{(2)}(\xi_5) - a^{\{0\}}_{(2)}(\xi_4) = \frac{2}{3} + \frac{2}{3} - \frac{1}{3} = 1,$$

and

$$a^{\{0\}}_{(3)}(\xi_5) + a^{\{0\}}_{(3)}(\xi_5) - a^{\{0\}}_{(3)}(\xi_4) = \frac{1}{2} + \frac{1}{2} - 0 = 1.$$

Using (2.1), we need only calculate $(\eta \star \eta)|_{\{0\}}$, where the product is

$$\eta|_{\{0\}} \cdot \eta|_{\{0\}} \prod_{i=1}^{3} e(\mathbb{C}_{(i)})^{a_{(i)}^{\{0\}}(\xi_5)+a_{(i)}^{\{0\}}(\xi_5)-a_{(i)}^{\{0\}}(\xi_4)} = 1 \cdot 1 \cdot (u)^1 \cdot (2u)^1 \cdot (3u)^1$$

for $\{0\}$, the fixed point of $Y^{\zeta_5\zeta_5} = Y^{\zeta_4}$. Thus $\eta \star \eta = 6u^3\delta$. Similarly, we compute the other products (see Table 9.2).

Table 9.2. The products of generators of $\mathrm{NH}_{S^1}^{*,\Gamma}(Y)$

$\star$	α	β	γ	δ	η
α	$3u\beta$	$2u\gamma$	$3u\delta$	η	$6u^3$
β		$2u\delta$	η	$2u^2$	$2u^2\alpha$
γ			$3u^2$	$u\alpha$	$3u^2\beta$
δ				$u\beta$	$2u^2\gamma$
η					$6u^3\delta$

Thus, as a ring,

$$\mathrm{NH}_{S^1}^{*,\Gamma}(Y;\mathbb{Z}) \cong \frac{\mathbb{Z}[u, \alpha, \beta, \gamma, \delta, \eta]}{I},$$

where I is the ideal generated by the product relations in Table 9.2.

Finally, to compute $H_{\mathrm{CR}}^*(Y//S^1;\mathbb{Z})$, we compute the kernel of the Kirwan map. Following [K], this kernel is

$$\ker(\kappa) = \langle 6u^3, \alpha, 3u\beta, 2u\gamma, 3u\delta, \eta\rangle. \tag{9.8}$$

Thus

$$H_{\mathrm{CR}}^*(Y_\Delta;\mathbb{Z}) \cong \frac{\mathbb{Z}[u, \alpha, \beta, \gamma, \delta, \eta]}{J},$$

where J is the ideal generated by the relations from Table 9.2 and (9.8).

It is sometimes more convenient to compute with the product $\smile$. We conclude this example with the computation of $\eta \smile \eta$. Using Table 9.1, we note that $e_1^*(\eta) \cdot e_2^*(\eta)$ is in $H_T^*(Y^{\zeta_5,\zeta_5})$. We notice that $Y^{\zeta_5,\zeta_5} = \{0\}$ and that in the cohomology of Y^{ζ_5,ζ_5}, $e_1^*(\eta) \cdot e_2^*(\eta) = 1$. Thus $\nu Y^{\zeta_5,\zeta_5} = \mathbb{C}_{(1)} \oplus \mathbb{C}_{(2)} \oplus \mathbb{C}_{(3)}$. We now check to see which of these lines are in the obstruction bundle by computing

$$a_{(1)}^{\{0\}}(\xi_5) + a_{(1)}^{\{0\}}(\xi_5) + a_{(1)}^{\{0\}}(\xi_4) = \frac{5}{6} + \frac{5}{6} + \frac{1}{3} = 2,$$

$$a_{(2)}^{\{0\}}(\xi_5) + a_{(2)}^{\{0\}}(\xi_5) + a_{(2)}^{\{0\}}(\xi_4) = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2,$$

and

$$a_{(3)}^{\{0\}}(\xi_5) + a_{(3)}^{\{0\}}(\xi_5) + a_{(3)}^{\{0\}}(\xi_4) = \frac{1}{2} + \frac{1}{2} + 0 = 1.$$

Thus the obstruction bundle is $E|_{Y^{\zeta_5,\zeta_5}} = \mathbb{C}_{(1)} \oplus \mathbb{C}_{(2)}$, and so the virtual class in this case is $\epsilon = 2u^2$. Finally, we note that the pushforward $(\overline{e}_3)_*$ multiplies the class $e_1^*(\eta) \cdot e_2^*(\eta) \cdot \epsilon$ by $3u\delta$, which is the Euler class of the normal bundle to Y^{ζ_2}. Thus $\eta \smile \eta = 6u^3\delta$. The other pairwise products can be computed easily in the same fashion, yielding, of course, the same multiplication table as in Table 9.2.

9.4. Relation to toric Deligne-Mumford stacks

Borisov, Chen, and Smith [BCS] compute the rational Chen-Ruan Chow ring of a toric Deligne-Mumford stack. This stack is determined by a combinatorial object called a *stacky fan* Σ. To each labeled polytope Δ, we may associate such a stacky fan. If the labels of the polytope are all 1, then the associated fan is the *canonical stacky fan* of [BCS]. Just as for smooth toric varieties, however, there are stacky fans that cannot be defined via labeled polytopes. Here we show that the [BCS] description of the Chow ring of these stacky fans corresponds to our description over $\mathbb{Q}$. While our results are less general in that we cannot describe the ring for all stacky fans, we have the advantage that in some cases, we can do computations over $\mathbb{Z}$. When Y is a symplectic toric orbifold, our results agree.

THEOREM 9.7

Let $Y = \mathbb{C}^N//K$ be a toric orbifold that can be realized as a Deligne-Mumford stack $\mathscr{X}(\Sigma)$. Then there is a ring isomorphism

$$H^*_{\mathrm{CR}}(Y) \cong \frac{\mathrm{NH}_K^{*,\diamond}(\mathbb{C}^N;\mathbb{Q})}{\ker \kappa_{\mathrm{NH}}} \longrightarrow A^*_{\mathrm{CR}}\big(\mathscr{X}(\Sigma)\big) \tag{9.9}$$

which divides all degrees in half.

Proof

We first construct the isomorphism of the modules. In what follows, all coefficients are taken to be $\mathbb{Q}$. As in Section 9.1, consider the short exact sequence of groups

$$1 \to K \to \mathbb{T}^N \to T \to 1$$

and the corresponding short exact sequence of dual Lie algebras

$$0 \longrightarrow \mathfrak{t}^* \xrightarrow{\varpi^*} (\mathbb{R}^N)^* \xrightarrow{j^*} \mathfrak{k}^* \longrightarrow 0.$$

Choose $\Phi : \mathbb{C}^N \to \mathfrak{k}^*$, a moment map with $Y = \Phi^{-1}(0)/K$.

The inertia stack defined in [BCS] is given as a quotient of a space $\mathscr{Z}$ given as the complement of the subspace arrangement. Our moment map level set is homotopy equivalent to their $\mathscr{Z}$. Indeed, $\mathscr{Z}$ is endowed with an algebraic $(K_{\mathbb{C}} \cong (\mathbb{C}^*)^k)$-action, and the $\mathbb{R}^k$ acts on our moment map level set by scaling inside $\mathscr{Z}$.

Borisov, Chen, and Smith then show that the inertia stack is

$$\coprod_{g\in\gamma}[\mathscr{Z}^g/K_{\mathbb{C}}],$$

where the disjoint union is taken over the set γ of finite stabilizers $g \in K_{\mathbb{C}}$. As such, they give a module isomorphism

$$A^*_{\mathrm{CR}}\big(\mathscr{X}(\Sigma)\big) \cong A^*\Big(\coprod_{g\in\gamma}[\mathscr{Z}^g/K_{\mathbb{C}}]\Big).$$

Each piece $[\mathscr{Z}^g/K_{\mathbb{C}}]$ of the inertia stack is again a toric orbifold. We now construct an isomorphism

$$\mathrm{NH}_K^{*;g}(\mathbb{C}^N) \cong \frac{H_K^*((\mathbb{C}^N)^g)}{\ker\kappa} \longrightarrow A^*([\mathscr{Z}^g/K_{\mathbb{C}}])$$

which divides degrees in half.

Borisov, Chen, and Smith describe $A^*([\mathscr{Z}^g/K_{\mathbb{C}}])$ as in Danilov [D], also referred to as the Stanley-Reisner description. For notational simplicity, we explain the case when $g = \mathrm{id}$. The remaining pieces of the module isomorphism are derived in the same way, using subgroups of T, $\mathbb{T}$, and K and subpolytopes of Δ as appropriate. We know that $[\mathscr{Z}/K_{\mathbb{C}}]$ is a T-toric variety with moment polytope Δ. Danilov proved that

$$A^*([\mathscr{Z}/K_{\mathbb{C}}]) \cong \frac{\mathbb{Q}[x_1,\ldots,x_N]}{(\mathscr{I},\mathscr{J})},$$

where $\deg(x_i) = 1$; $\mathscr{I}$ is the ideal generated by $\prod_{i\in I} x_i$ for all $I \subset \{1,\ldots,N\}$ such that the I-facets do not intersect in Δ; and $\mathscr{J} = \big\{\sum \alpha_i x_i \;\big|\; \alpha \in \varpi^*(\mathfrak{t})^*\big\}$.

On the other hand, following [TW], we describe $H_K^*(\mathbb{C}^N)/\ker\kappa$. There is a commutative diagram

$$\begin{tikzcd}
H_{\mathbb{T}}^*(\mathbb{C}^N) \arrow[r, "r_K^*"] \arrow[d, two heads, "\kappa^{\mathbb{T}}"'] & H_K^*(\mathbb{C}^N) \arrow[d, two heads, "\kappa"] \\
H_{\mathbb{T}}^*\big(\Phi^{-1}(0)\big) \arrow[r, "r_K^*"'] & H_K^*\big(\Phi^{-1}(0)\big)
\end{tikzcd}$$

The map $\kappa^{\mathbb{T}}$ is a surjection due to an equivariant version of Theorem 6.1. The ring $H^*_{\mathbb{T}}(\mathbb{C}^N)$ is isomorphic to $\mathbb{Q}[x_1, \dots, x_N]$, where $\deg(x_i) = 2$. The kernel of the restriction map r_K^* is the ideal $\mathscr{J}$. As Tolman and Weitsman show explicitly in [TW, proof of Theorem 7], the kernel of κ is $\mathscr{I}$. Finally, since the K-action is locally free, we have an isomorphism $H^*_K(\Phi^{-1}(0)) \cong H^*([\Phi^{-1}(0)/K])$. Thus identifying the x_i's gives an isomorphism

$$H^*\big([\Phi^{-1}(0)/K]\big) \longrightarrow A^*([\mathscr{Z}/K_{\mathbb{C}}])$$

which divides degrees in half. Putting these together produces a module isomorphism

$$\frac{\mathrm{NH}^{*,\diamond}_K(\mathbb{C}^N;\mathbb{Q})}{\ker \kappa_{\mathrm{NH}}} \longrightarrow A^*_{\mathrm{CR}}\big(\mathscr{X}(\Sigma)\big).$$

More generally, we have a commutative diagram in inertial cohomology

$$\begin{tikzcd}
\mathrm{NH}^{*,\diamond}_{\mathbb{T}}(\mathbb{C}^N) \arrow[r] \arrow[d, two heads, "\kappa^{\mathbb{T}}_{\mathrm{NH}}"'] & \mathrm{NH}^{*,\diamond}_K(\mathbb{C}^N) \arrow[d, two heads, "\kappa_{\mathrm{NH}}"] & \mathrm{NH}^{*,\Gamma}_K(\mathbb{C}^N) \arrow[l, hook'] \arrow[dl, two heads] \\
\mathrm{NH}^{*,\diamond}_{\mathbb{T}}\big(\Phi^{-1}(0)\big) \arrow[r] & \mathrm{NH}^{*,\diamond}_K\big(\Phi^{-1}(0)\big) & \\
\mathrm{NH}^{*,\gamma}_{\mathbb{T}}\big(\Phi^{-1}(0)\big) \arrow[u, hook] \arrow[ur, two heads] & &
\end{tikzcd}$$

Borisov, Chen, and Smith start at $H^*_{\mathbb{T}}(\mathbb{C}^N)$, first restrict to the level set, and then restrict to the subgroup K. As a result, $\mathrm{NH}^{*,\gamma}_{\mathbb{T}}(\Phi^{-1}(0)) \cong \mathbb{Q}[x_1, \dots, x_N]/\mathscr{I}$ is precisely the numerator $\mathbb{Q}[N]^{\Sigma}$ in [BCS, Theorem 1.1], once degrees are divided in half.

That the module isomorphism is a ring isomorphism is a simple exercise in comparing the ring structures constructed here and in [BCS]. We leave the details to the reader. □

Finally, we mention the relationship with crepant resolutions. There is a general conjecture that the Chen-Ruan cohomology of an orbifold coincides with the ordinary cohomology (or Chow ring) of a crepant resolution, if one exists (see [CR]). In [BCS, Section 7], Borisov, Chen, and Smith find necessary and sufficient conditions for a toric orbifold Y to have a crepant resolution $\widetilde{Y}$. They describe a flat family $\mathscr{T} \to \mathbb{P}^1$ of rings such that $\mathscr{T}_0$ is the Chen-Ruan Chow ring of Y and $\mathscr{T}_\infty$ is the Chow ring of $\tilde{Y}$. Finally, they compute the Chen-Ruan Chow ring of the weighted projective space $Y = \mathbb{P}_{1,2,1}$ and the Chow ring of a crepant resolution $\tilde{Y}$ and show that they are not

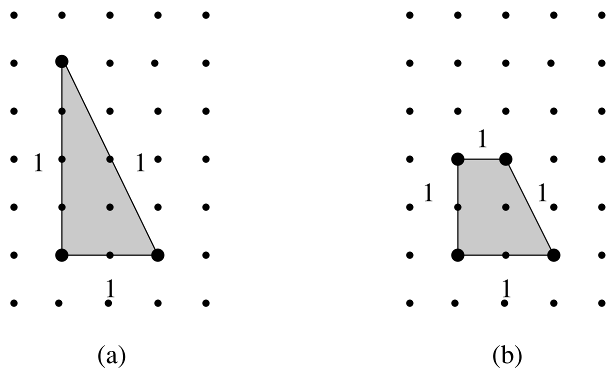


Figure 9.3. (a) shows the labeled polytope for $Y = \mathbb{P}_{1,2,1}$; (b) shows the labeled polytope for $\widetilde{Y}$, its crepant resolution

isomorphic as rings over $\mathbb{Q}$.

We show that even the module structures are not the same over $\mathbb{Z}$. The space $Y = \mathbb{P}_{1,2,1}$ and its crepant resolution $\widetilde{Y}$ have moment polytopes (see Figure 9.3). Using the methods that we have described, we may compute

$$H^*_{\mathrm{CR}}(\mathbb{P}^2_{1,2,1}; \mathbb{Z}) = \frac{\mathbb{Z}[u, \alpha]}{\langle u^2 - \alpha^2, 2u^3, 2u\alpha\rangle},$$

which yields

$$H^i_{\mathrm{CR}}(\mathbb{P}^2_{1,2,1}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, \\ \mathbb{Z} \oplus \mathbb{Z}, & i = 2, \\ \mathbb{Z} \oplus \mathbb{Z}_2, & i = 4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & i = 2n > 4, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we see that there is torsion in all higher degrees. This certainly does not happen for the crepant resolution, as it is a smooth variety. Thus, even as modules, the two cohomologies do not agree over $\mathbb{Z}$.

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