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# COHOMOLOGY OF SYMPLECTIC REDUCTIONS OF GENERIC COADJOINT ORBITS

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ABSTRACT. Let $\mathcal{O}_\lambda$ be a generic coadjoint orbit of a compact semi-simple Lie group $K$. Weight varieties are the symplectic reductions of $\mathcal{O}_\lambda$ by the maximal torus $T$ in $K$. We use a theorem of Tolman and Weitsman to compute the cohomology ring of these varieties. Our formula relies on a *Schubert basis* of the equivariant cohomology of $\mathcal{O}_\lambda$, and it makes explicit the dependence on $\lambda$ and a parameter in $Lie(T)^* =: \mathfrak{t}^*$.

## 1. INTRODUCTION

Let $K$ be a compact semisimple Lie group, $T \subset K$ a maximal torus and $\mathfrak{t} \subset \mathfrak{k}$ their Lie algebras. Pick a fundamental chamber in $\mathfrak{t}^*$ and a point $\lambda$ in the interior. Let $\mathcal{O}_\lambda$ be the orbit of $\lambda$ under the coadjoint representation of $K$ on $\mathfrak{k}^*$. $\mathcal{O}_\lambda$ is diffeomorphic to the flag variety $K/T$, and it has a naturally occurring symplectic form $\omega$ known as the Kirillov-Kostant-Souriau form. The action of $T$ on $\mathcal{O}_\lambda$ is Hamiltonian, which means that there is an invariant map

$$\Phi : \mathcal{O}_\lambda \to \mathfrak{t}^*$$

satisfying $\omega(X_\eta, \cdot) = d\Phi^\eta$, where $\eta \in \mathfrak{t}$, $X_\eta$ is the vector field on $\mathcal{O}_\lambda$ generated by $\eta$, and $\Phi^\eta(m) = \Phi(m)(\eta)$ is defined by the natural pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$. We call $\Phi$ a *moment map* for this action.

The image of $\Phi$ is the convex hull of $W \cdot \lambda$, the Weyl group orbit of $\lambda$. Let $\mu \in \Phi(\mathcal{O}_\lambda)$ be a regular value of $\Phi$. We define the *symplectic reduction* at $\mu$ by

$$\Phi^{-1}(\mu)/T = \mathcal{O}_\lambda//T(\mu).$$

The goal of this note is to give a presentation of the cohomology[1] ring of $\mathcal{O}_\lambda//T(\mu)$ in terms of the root system of $K$. We present $H^*(\mathcal{O}_\lambda//T(\mu))$ as a quotient of the $T$-equivariant cohomology ring $H^*_T(\mathcal{O}_\lambda)$ by a certain ideal. We rely on the following fundamental result.

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[1]Only cohomology with coefficients in the field $\mathbb{Q}$ of rational numbers will be considered throughout this paper.

**Theorem 1.1** (Kirwan). *Let $M$ be a compact symplectic manifold with a Hamiltonian $T$ action, where $T$ is a compact torus. If $\mu \in \mathfrak{t}^*$ is a regular value of $\Phi$, then the restriction map in equivariant cohomology,*

$$\kappa : H_T^*(M) \to H_T^*(\Phi^{-1}(\mu)),$$

*is surjective.*

Since the $T$ action is locally free on level sets of the moment map at regular values, $H_T^*(\Phi^{-1}(\mu)) = H^*(M//T(\mu))$. The resulting map $\kappa : H_T^*(M) \to H^*(M//T(\mu))$ is called the *Kirwan map*. Kirwan's result is of particular importance because the equivariant cohomology can be described in terms of the equivariant cohomology of the fixed point sets of the $T$ action. In the case of isolated fixed points, this is just a sum of polynomial rings.

**Theorem 1.2** (Kirwan). *Let $M$ be a compact Hamiltonian $T$-space. Let $M^T$ denote the fixed point set of the $T$ action. The restriction map*

$$i^* : H_T^*(M) \to H_T^*(M^T)$$

*is injective. In the case that $M^T$ is a finite set of points,*

$$H_T^*(M^T) = \bigoplus_{p \in M^T} \mathbb{Q}[x_1, \ldots, x_n]$$

*where $n = \dim T$.*

A presentation of the cohomology ring of the reduced space $M//T(\mu)$ can be obtained by using the following description of the kernel of the Kirwan map, which is due to Tolman and Weitsman [TW]. If $\alpha$ is in $H_T^*(M)$ we denote

$$supp(\alpha) = \{p \in M^T : \alpha|_p \neq 0\}.$$

Fix an arbitrary inner product $\langle\ ,\ \rangle$ on $\mathfrak{t}^*$.

**Theorem 1.3** (Tolman-Weitsman). *The kernel of the Kirwan map $\kappa$ is the ideal of $H_T^*(M)$ generated by all $\alpha \in H_T^*(M)$ with the property that there exists $\xi \in \mathfrak{t}^*$ such that*

$$\Phi(supp(\alpha)) \subset \{x \in \mathfrak{t}^* | \langle \xi, x\rangle \leq \langle \xi, \mu\rangle\}.$$

*In other words, $\ker \kappa$ consists of sums of equivariant cohomology classes $\alpha$ with the property that all points of $supp(\alpha)$ are mapped by $\Phi$ to the same side of an affine hyperplane in $\mathfrak{t}^*$ that passes through $\mu$.*

The $T$-equivariant cohomology ring of the coadjoint orbit $\mathcal{O}_\lambda = K/T$ is well understood. Kostant and Kumar constructed in [KK] a basis $\{x_w\}_{w \in W}$ of $H_T^*(K/T)$ as a $H_T^*(pt)$-module, which we refer to as the *Schubert basis*. Let $B$ be a Borel subgroup in $G := K^{\mathbb{C}}$, and let $B_-$ be the opposite Borel subgroup. For any $v \in W$, let $X_v = \overline{B_- \tilde{v} B}/B$, where $\tilde{v}$ is any choice of lift of $v \in W$ in the normalizer of the torus. These *opposite Schubert varieties* are $T$-invariant subvarieties of $G/B \cong K/T$. The basis $\{x_w\}$ is uniquely defined by the property that

$$\int_{X_v} x_w = \delta_{vw}.$$

Theorem 1.2 suggests the importance of knowing how to restrict the classes $x_w$ to fixed points $W \cdot \lambda$. This formula was worked out for general $K$ by S. Billey [Bi]. In particular, it is easy to show that $x_w|_v = 0$ if $v \not\leq w$ in the Bruhat order.[2] In other words,

$$supp(x_w) = \{v\lambda : v \leq w\}.$$

To each $\tau \in W$ we can associate the new basis

$$\{x_w^\tau = \tau \cdot x_{\tau^{-1}w}\}_{w \in W},$$

whose elements have the property

$$supp(x_w^\tau) = \{v\lambda : \tau^{-1}v \leq \tau^{-1}w\}.$$

Let $\lambda_1, \ldots, \lambda_l \in \mathfrak{t}^*$ denote the fundamental weights associated to the chosen fundamental chamber of $\mathfrak{t}^*$. Let $\langle\ ,\ \rangle$ be the restriction to $\mathfrak{t}^*$ of a $K$-invariant product on $\mathfrak{k}^*$. Our main result is:

**Theorem 1.4.** *The cohomology ring $H^*(\mathcal{O}_\lambda//T(\mu))$ is isomorphic to the quotient of $H_T^*(K/T)$ by the ideal generated by*

$$\{x_v^\tau : \textit{there exists } j \textit{ such that } \langle \lambda_j, \tau^{-1}v\lambda\rangle \leq \langle \lambda_j, \tau^{-1}\mu\rangle\}.$$

*Remark* 1. One can take the description of $H_T^*(K/T)$ (see, for instance, [Br]) and deduce a precise presentation of the cohomology ring $H^*(\mathcal{O}_\lambda//T(\mu))$ in terms of generators and relations.

*Remark* 2. For $K = SU(n)$ this result was proven by the first author in [Go1].

## 2. Primary description of $\ker \kappa$

For any $\xi \in \mathfrak{t}^*$ we denote by $f_\xi$ the corresponding height function on $\mathcal{O}_\lambda$,

$$f_\xi(x) = \langle \xi, x\rangle.$$

Under the pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$, the function $f_\xi$ is a component of the moment map. In fact, it is well known that $f_\xi$ is Morse-Bott for all $\xi \in \mathfrak{t}^*$. Denote by $C \subset \mathfrak{t}^*$ the fundamental (positive) Weyl chamber, which can be described by

$$C = \{r_1\lambda_1 + \cdots + r_l\lambda_l : \text{ all } \ r_j > 0\},$$

and let $\overline{C}$ be its closure.

**Lemma 2.1.** *Let $\tau$ be in $W$ and $\xi$ in $\tau\overline{C}$. If $\tau^{-1}v < \tau^{-1}w$ in the Bruhat order, then $f_\xi(v\lambda) \leq f_\xi(w\lambda)$.*

*Proof.* The result follows immediately from the fact that if $\xi \in C$, then the unstable manifold of $f_\xi$ through $v\lambda$ with respect to the Kähler metric on

$$\mathcal{O}_\lambda = K/T = G/B$$

is just the Bruhat cell $B \cdot vB/B$ (see, for instance, [Ko]). □

[2] The class $x_w$ differs from the $\xi^w$ constructed in [KK] by the relationship $x_w := w_0 \cdot \xi^{w_0 w}$, where $w_0$ is the longest element of $W$.

The main result of this section is:

**Theorem 2.1.** *Suppose that $\alpha \in H_T^*(\mathcal{O}_\lambda)$ has the property that*

$$\Phi(supp(\alpha)) \subset \{x \in \mathfrak{t}^* : \langle \xi, x\rangle \leq \langle \xi, \mu\rangle\}.$$

*Then $\alpha$ can be decomposed as*

$$\alpha = \sum_{w \in W} a_w^\tau x_w^\tau$$

*with $a_w^\tau \in H_T^*(pt)$, such that if $a_w^\tau \neq 0$, then*

$$\Phi(supp(x_w^\tau)) \subset \{x \in \mathfrak{t}^* : \langle \xi, x\rangle \leq \langle \xi, \mu\rangle\}.$$

*Proof.* Take $\tau \in W$ such that $\xi \in \tau\overline{C}$. Suppose that the decomposition of $\alpha$ with respect to the basis $\{x_w^\tau\}_{w\in W}$ is of the form

$$\alpha = \sum_{w \in W} a_w^\tau x_w^\tau + a_{v_1}^\tau x_{v_1}^\tau + \cdots + a_{v_r}^\tau x_{v_r}^\tau, \tag{1}$$

where the first sum contains only $w$ with

$$\langle \xi, w\lambda\rangle \leq \langle \xi, \mu\rangle,$$

whereas

$$\langle \xi, v_j\lambda\rangle > \langle \xi, \mu\rangle, \quad a_{v_j}^\tau \in S(\mathfrak{t}^*), a_{v_j}^\tau \neq 0,$$

for any $1 \leq j \leq l$. We may assume that $v_1$ has the property that there exists no $j > 1$ with $\tau^{-1}v_1 < \tau^{-1}v_j$. Now let us evaluate both sides of (1) at $v_1\lambda$. Since

$$\langle \xi, w\lambda\rangle \leq \langle \xi, \mu\rangle < \langle \xi, v_1\lambda\rangle,$$

by Lemma 2.1 we must have

$$x_w^\tau|_{v_1\lambda} = 0$$

for any $w$ corresponding to a term in the first sum in (1). It follows that

$$\alpha|_{v_1\lambda} = a_{v_1}^\tau x_{v_1}^\tau|_{v_1\lambda} \neq 0;$$

so $v_1\lambda$ is in $supp(\alpha)$ even though $\langle \xi, v_1\lambda\rangle > \langle \xi, \mu\rangle$. This is a contradiction. □

## 3. Proof of the main result

We now prove Theorem 1.4. Let $v$ and $\tau$ in $W$ be such that

$$\langle \lambda_j, \tau^{-1}v\lambda\rangle \leq \langle \lambda_j, \tau^{-1}\mu\rangle, \tag{2}$$

for some $1 \leq j \leq l$. We show that $x_v^\tau$ is in the kernel of the Kirwan map

$$\kappa : H_T^*(\mathcal{O}_\lambda) \to H^*(\mathcal{O}_\lambda//T(\mu)).$$

Let $\xi = \tau\lambda_j$ be in $\tau\overline{C}$. Note that if $w \in supp(x_v^\tau)$, then $\tau^{-1}w \leq \tau^{-1}v$ implies by Lemma 2.1 that

$$\langle \xi, w\lambda\rangle \leq \langle \xi, v\lambda\rangle \leq \langle \xi, \mu\rangle.$$

Thus $x_v^\tau \in \ker \kappa$.

Now let us consider $\alpha \in H_T^*(K/T)$ with the property that there exists $\xi \in \mathfrak{t}^*$ with

$$supp(\alpha) \subset \{x \in \mathfrak{t}^* | \langle \xi, x\rangle \leq \langle \xi, \mu\rangle\}.$$

Take $\tau \in W$ such that $\xi \in \tau\overline{C}$. By Theorem 2.2, we can decompose $\alpha$ as

$$\alpha = \sum_{w \in W} a_w^\tau x_w^\tau \tag{3}$$

where $a_w^\tau$ can be nonzero only if

$$supp(x_w^\tau) \subset \{x \in \mathfrak{t}^* | \langle \xi, x \rangle \leq \langle \xi, \mu \rangle\}.$$

In particular, if $a_w^\tau \neq 0$, then

$$\langle \xi, w\lambda \rangle \leq \langle \xi, \mu \rangle. \tag{4}$$

Since $\xi$ is in $\tau\overline{C}$, it can be written as

$$\xi = \tau \sum_{j=1}^{l} r_j \lambda_j, \tag{5}$$

where all $r_j$ are nonnegative. So (4) and (5) imply that there exists $j \in \{1, \ldots, l\}$ such that

$$\langle \tau\lambda_j, w\lambda \rangle \leq \langle \tau\lambda_j, \mu \rangle.$$

In other words, each nonzero term in the right-hand side of (3) is a multiple of a $x_w^\tau$ of the type claimed in Theorem 1.4. □

*Remark.* It follows that, in the particular situation of generic coadjoint orbits, in order to cover the whole Tolman-Weitsman kernel of the Kirwan map it is sufficient to consider affine hyperplanes through $\mu$ that are perpendicular to vectors of the type $\tau\lambda_j$, with $\tau \in W$ and $j \in \{1, \ldots, l\}$. But these are just the hyperplanes parallel to the walls of the moment polytope. This result concerning a "sufficient set of hyperplanes" has been proved by the first author in [Go2], for an *arbitrary* Hamiltonian torus action on a compact manifold.

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