

## AN EFFECTIVE ALGORITHM FOR THE COHOMOLOGY RING OF SYMPLECTIC REDUCTIONS

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### Abstract

Let  $G$  be a compact torus acting on a compact symplectic manifold  $M$  in a Hamiltonian fashion, and  $T$  a subtorus of  $G$ . We prove that the kernel of  $\kappa : H_G^*(M) \rightarrow H^*(M//G)$  is generated by a small number of classes  $\alpha \in H_G^*(M)$  satisfying very explicit restriction properties. Our main tool is the *equivariant Kirwan map*, a natural map from the  $G$ -equivariant cohomology of  $M$  to the  $G/T$ -equivariant cohomology of the symplectic reduction of  $M$  by  $T$ . We show this map is surjective. This is an equivariant version of the well-known result that the (nonequivariant) Kirwan map  $\kappa : H_G^*(M) \rightarrow H^*(M//G)$  is surjective. We also compute the kernel of the equivariant Kirwan map, generalizing the result due to Tolman and Weitsman [TW] in the case  $T = G$  and allowing us to apply their methods inductively. This result is new even in the case that  $\dim T = 1$ . We close with a worked example: the cohomology ring of the product of two  $\mathbb{C}P^2$ s, quotiented by the diagonal 2-torus action.

### 1 Introduction and Statement of Results

Let  $(M, \omega)$  be a symplectic manifold with an action of a compact torus  $G$ . A *moment map* is an invariant map

$$\Phi : M \longrightarrow \mathfrak{g}^*$$

which intertwines the group action and the symplectic form by the *moment map condition*

$$\omega(\cdot, X_\xi) = d\langle \Phi, \xi \rangle, \tag{1}$$

where  $\xi \in \mathfrak{g}$ ,  $X_\xi$  is the vector field on  $M$  generated by  $\xi$ , and  $\langle \cdot, \cdot \rangle$  is the pairing of  $\mathfrak{g}^*$  with  $\mathfrak{g}$ . We also write  $\Phi^\xi$  to indicate  $\langle \Phi, \xi \rangle$ , the  $\xi$ -component of  $\Phi$ . Condition (1) and the nondegeneracy of  $\omega$  imply that singular points of  $\Phi$  occur when  $X_\xi = 0$ , or when a subtorus of  $G$  acts trivially. If  $C \subset M$  is a component of the fixed point set of  $G$  on  $M$ , then  $\Phi(C)$  has constant value in  $\mathfrak{g}^*$ .

For  $M$  a compact manifold, the image of  $\Phi$  is a convex polytope in  $\mathfrak{g}^*$  [GuS]. Let  $d = \dim G$ . For any subtorus  $T \subset G$ , denote by  $M^T$  the fixed point set of  $M$  under  $T$ . We say  $T$  is *generic* if  $M^T = M^G$ . The terminology is appropriate as there are a finite number of subtori  $T$  such that  $M^T$  contains but does not equal  $M^G$ . If  $T$  is generic,  $\Phi(M^T)$  is a set of isolated points. In general, however, the images  $\Phi(M^T)$  form *codimension- $k$  walls* of the polytope, where  $d-k$  is the dimension of the subtorus of  $G/T$  acting effectively. These walls may be internal to the polytope. Let  $S^1 \subset G$  be a 1-dimensional subtorus. If a component of  $M^{S^1}$  has an effective  $G/S^1$  action, then its image will be a codimension-1 wall. These include but are not restricted to the facets of the polytope.

When  $\mu$  is a regular value of  $\Phi$ ,  $\Phi^{-1}(\mu)$  is a submanifold of  $M$  and has a locally free  $G$  action by the invariance of  $\Phi$ . The quotient space  $\Phi^{-1}(\mu)/G$  is called the *symplectic reduction* and is denoted  $M//G(\mu)$ , where the parameter  $\mu$  is suppressed when  $\mu = 0$ .

This paper is concerned with the ordinary, rational cohomology of symplectic reductions and its relationship to the  $G$ -equivariant cohomology of  $M$ . By definition, the  $G$ -equivariant cohomology of  $M$  is

$$H_G^*(M) := H^*(M \times_G EG)$$

where  $EG$  is a contractible space with a free  $G$  action and  $M \times_G EG$  indicates the product  $M \times EG$  quotiented by the diagonal  $G$  action. For more details, see [AB2].

When  $G$  acts locally freely on a manifold  $Z$ , the  $G$ -equivariant cohomology of  $Z$  is the ordinary cohomology of the quotient  $Z/G$ . In particular for  $\mu$  a regular value of the moment map,

$$H_G^*(\Phi^{-1}(\mu)) = H^*(M//G(\mu)).$$

**Theorem 1.1** (Kirwan). *Let  $M$  be a compact symplectic manifold with a Hamiltonian  $G$  action, where  $G$  is a compact torus. Let  $\Phi$  be a moment map for the  $G$  action on  $M$ . For any regular value  $\mu \in \mathfrak{g}^*$ , the natural map*

$$\kappa : H_G^*(M) \longrightarrow H^*(M//G(\mu))$$

*induced from the inclusion  $\Phi^{-1}(\mu) \subset M$  is a surjection.*

We generalize this theorem to the following equivariant version.

**Theorem 1.2.** *Let  $M$  be a compact symplectic manifold with a Hamiltonian  $G$  action, where  $G$  is a compact torus and  $T$  a subtorus, not necessarily generic. Let  $\Phi_G$  and  $\Phi_T$  be moment maps for the respective actions. For a regular value  $\mu \in \mathfrak{t}^*$  of  $\Phi_T$ , the inclusion of the submanifold  $\Phi_T^{-1}(\mu)$  of  $M$*

induces a surjection in equivariant cohomology

$$\kappa_T : H_G^*(M) \longrightarrow H_{G/T}^*(M//T(\mu)).$$

Note that we use  $\kappa$  to indicate the map on cohomology induced by the restriction from  $M$  to  $\Phi_G^{-1}(0)$ , and  $\kappa_T$  for the map on cohomology induced by the restriction from  $M$  to  $\Phi_T^{-1}(0) \supseteq \Phi_G^{-1}(0)$ .

These two theorems allow one to compute the cohomology of the symplectic reduction by a torus action, as long as one can compute the equivariant cohomology and the kernel of this map. In certain cases, the equivariant cohomology has been completely described [GoKM], [GH]. The kernel of the Kirwan map in Theorem 1.1 has also been described in very general terms, which we present below as Theorem 1.4. These results hinge on the key fact that for compact Hamiltonian torus spaces, the equivariant cohomology injects into the equivariant cohomology of the fixed point set under the natural restriction map [K].

**Theorem 1.3** (Kirwan). *Let  $M$  be a compact Hamiltonian  $G$  space for  $G$  a compact torus. Let  $\mathcal{C}$  be the collection of connected components of the fixed point set  $M^G$ . Then*

$$H_G^*(M) \hookrightarrow H_G^*(M^G) = \bigoplus_{C \in \mathcal{C}} H_G^*(pt) \otimes H^*(C) \tag{2}$$

is an injection. For a class  $\alpha \in H_G^*(M)$  and  $C \in \mathcal{C}$ , we write  $\alpha|_C$  to indicate the restriction to the fixed component  $C$ .

The relationship between surjectivity onto the symplectic reduction and injectivity of the equivariant cohomology into that of the fixed point set is manifest in the following description of  $\ker \kappa$  from Theorem 1.1.

**Theorem 1.4** (Tolman-Weitsman). *Let  $M$  be a compact Hamiltonian  $G$ -space, with moment map  $\Phi$ . Let  $\mu \in \mathfrak{g}^*$  be a regular value of  $\Phi$ . The kernel of the Kirwan map*

$$\kappa : H_G^*(M) \longrightarrow H^*(M//G(\mu))$$

is the ideal  $\langle K_G^{\mathfrak{g}}(\mu) \rangle$  in  $H_G^*(M)$  generated by  $K_G^{\mathfrak{g}}(\mu) = \bigcup_{\xi \in \mathfrak{g}} K_G^{\xi}(\mu)$  where

$$K_G^{\xi}(\mu) := \left\{ \alpha \in H_G^*(M) \mid \alpha|_C = 0 \text{ for all connected components } C \text{ of } M^G \right. \\ \left. \text{with } \langle \Phi_G(C), \xi \rangle > \langle \mu, \xi \rangle \right\}.$$

We prove an equivariant analogue of this theorem:

**Theorem 1.5.** *Let  $M$  be a compact Hamiltonian  $G$  space with moment map  $\Phi_G$ . Let  $T$  be any subtorus of  $G$  and  $\Phi_T$  the corresponding moment*

map for the  $T$  action. For  $\mu$  be a regular value of  $\Phi_T$ , the kernel of the equivariant Kirwan map

$$\kappa_T : H_G^*(M) \longrightarrow H_{G/T}^*(M//T(\mu))$$

is the ideal  $\langle K_G^{\mathfrak{t}}(\mu) \rangle$  generated by  $K_G^{\mathfrak{t}}(\mu) = \bigcup_{\xi \in \mathfrak{t}} K_G^{\xi}(\mu)$  where  $K_G^{\xi}(\mu)$  is defined as in Theorem 1.4.

The subindex  $G$  on  $K_G^{\mathfrak{g}}(\mu)$  and  $K_G^{\mathfrak{t}}(\mu)$  indicates that these generate ideals in  $H_G^*(M)$ , while the superindices indicate the relevant set of vectors in the theorems. We will use  $\langle K_G^{\mathfrak{t}}(\mu) \rangle$  to indicate the ideal in  $H_G^*(M)$  generated by  $K_G^{\mathfrak{t}}(\mu)$  and we will suppress the parameter  $\mu$  when  $\mu = 0$ . Notice that the difference between the kernel of  $\kappa$  and that of  $\kappa_T$  is that the union in Theorem 1.5 is taken only over  $\xi \in \mathfrak{t}$ . The significance is that Theorem 1.4 can be recovered by the successive application of Theorem 1.5 to one-dimensional subtori of  $G$ . In the case that  $\mu = 0$ , for each  $S^1 \subset G$ , the kernel is generated by  $K_{S^1}^{\xi}$  and  $K_{S^1}^{-\xi}$  for a choice of generator  $\xi \in \mathfrak{s}^1$ . It follows that the kernel of  $\kappa$  is generated by classes  $\alpha \in H_G^*(M)$  satisfying one of  $d$  conditions, where  $d = \dim G$ .

The final contribution of this article is to find a small set  $\Xi$  of  $\xi \in \mathfrak{g}$  such that  $K_G^{\mathfrak{g}}(\mu)$  in Theorem 1.4 can be replaced by  $K_G^{\Xi}(\mu) := \bigcup_{\xi \in \Xi} K_G^{\xi}(\mu)$ . Let  $\Xi \subset \mathfrak{g}$  be the (finite) set of unit vectors perpendicular to codimension-1 walls of the moment polytope. For any  $\xi \in \Xi$ , the annihilator  $\xi^\perp$  is a hyperplane in  $\mathfrak{g}^*$  through 0 and parallel to a codimension-1 wall of the moment polytope. These hyperplanes, shifted to pass through  $\mu$ , are the key to the kernel of  $\kappa$ .

**Theorem 1.6.** *The kernel of  $\kappa : H_G^*(M) \rightarrow H^*(M//G(\mu))$  is generated by classes  $\alpha \in H_G^*(M)$  with the following property. There exists an oriented hyperplane  $H_\mu^\alpha$  through  $\mu$  and parallel to a codimension-1 wall of the moment polytope such that  $\alpha$  restricts to 0 on the set of all fixed points whose image under  $\Phi$  lie to the positive side of  $H_\mu^\alpha$ .*

We prove this theorem in section 5.

## 2 Equivariant Morse Theory

First we state several basic facts about equivariant Morse theory (as developed in [AB1]). We then refine these ideas to gain equivariant homotopy information that is standard in the case that the Morse function has only isolated critical points and there is no group action.

Let  $f$  be a smooth function on a compact manifold  $M$  and  $C$  a connected

component of the critical set of  $f$  on  $M$ . Choose a Riemannian metric on  $M$ . We say  $C$  is a *non-degenerate critical manifold* for  $f$  if

1.  $C \subset M$  is a submanifold of  $M$  such that  $df = 0$  along  $C$ , and
2. The Hessian  $H_C f$  (the matrix of second derivatives of  $f$ ) is non-degenerate on the normal bundle  $\nu C$  of  $C$  in  $M$ .

If every connected component of the critical set is non-degenerate, we say  $f$  is *Morse–Bott*. At every non-degenerate critical submanifold, we use the Riemannian metric to identify a neighborhood of the zero-section in the normal bundle  $\nu C$  with a tubular neighborhood of  $C$  in  $M$ . The Hessian defines a splitting  $\nu C = \nu^+ C \oplus \nu^- C$  into positive and negative normal bundles. The dimension of the fibres of  $\nu^- C$  is called the *Morse index* of  $C$  and is denoted  $\lambda_C$ .

Now assume that  $M$  has a  $G$  action, where  $G$  is a compact torus, and that the metric and the function  $f$  are invariant with respect to this action. The splitting of  $\nu C$  into positive and negative bundles is equivariant. This setting mimics that in which  $f$  is a generic component of the moment map, and  $C$  is a connected component of  $M^G$ .

The proofs of the following two lemma are close to identical to those presented by Milnor [M] in the case that critical manifolds are points and there is no group action. We note only the minor differences in this more general setting. To make the lemmas true for Morse–Bott functions, all local calculations must include extra coordinates along the critical manifolds. To make the lemmas equivariant, we choose an invariant Riemannian metric (so that the group acts by isometries) and we equivariantly identify a neighborhood of the zero-section in the normal bundle of a critical submanifold with a tubular neighborhood of that submanifold (see [Au]). This makes all relevant maps and homotopies equivariant. For Lemmas 2.1 and 2.2, assume  $M$  is a compact manifold with a  $G$  action and a  $G$ -invariant Riemannian metric, where  $G$  is a compact torus. Let  $f : M \rightarrow \mathbb{R}$  be an invariant Morse–Bott function. Let  $M^a = f^{-1}(-\infty, a]$  for  $a \in \mathbb{R}$ .

LEMMA 2.1. *Suppose  $a < b$  and  $f^{-1}[a, b]$  is compact and contains no critical points of  $f$ . Then*

$$M^a \simeq_G M^b$$

*is a  $G$ -equivariant homotopy equivalence.*

LEMMA 2.2. *Suppose there are  $m$  connected components  $C_1, \dots, C_m$  of the critical set of  $f$  with the same critical value, and such that  $f(C_1) = \dots = f(C_m) \in (a, b)$ . Then  $M^b$  is equivariantly homotopic to  $M^a$  with a*

$\lambda_i$ -cell bundle over  $C_i$  attached for each  $i = 1, \dots, m$ , where  $\lambda_i$  is the Morse index of  $C_i$ .

These homotopy theorems lead the following results about equivariant cohomology. For the sake of simplicity, we assume that  $f$  has distinct values for distinct connected components of the critical set. We note that the theorems proven below can be generalized (using Lemma 2.2) to include the case that  $f$  does not have distinct values for distinct critical sets, however the notation becomes cumbersome.

Assume now that the  $G$  action is Hamiltonian, and that  $f$  is a component of the moment map for the  $G$  action. As mentioned, we assume also that  $f$  separates the critical sets, i.e. one can order the critical sets  $C_0, C_1, \dots, C_k$  of  $f$  so that  $f(C_i) < f(C_j)$  if and only if  $i < j$ . If  $f$  is generic, these critical sets are the fixed points of the  $G$  action.

The fundamental principle introduced by Atiyah and Bott [AB1] is that **Theorem 2.1.** *An equivariant cohomology class on  $C_0$  extends to a class on  $M^a$  for any  $a \in \mathbb{R}$ ; in particular, it extends to one on all of  $M$  (although not uniquely).*

We prove this theorem following [AB1] and [TW].

*Proof.* Define the sets

$$M_i^+ := f^{-1}(-\infty, f(C_i) + \epsilon_i) \quad \text{and} \tag{3}$$

$$M_i^- := f^{-1}(-\infty, f(C_i) - \epsilon_i) \tag{4}$$

where  $\epsilon_i > 0$  is small enough that  $C_i$  is the only critical set in  $f^{-1}(f(C_i) - \epsilon_i, f(C_i) + \epsilon_i)$ . Note that by Lemma 2.1

$$M_i^+ \simeq_G M_{i+1}^- \tag{5}$$

For each  $i$ , there is a long exact sequence in  $G$ -equivariant cohomology

$$\dots \rightarrow H_G^*(M_i^+, M_i^-) \rightarrow H_G^*(M_i^+) \rightarrow H_G^*(M_i^-) \rightarrow H_G^{*+1}(M_i^+, M_i^-) \rightarrow \dots \tag{6}$$

As before, we equivariantly identify a tubular neighborhood of  $C_i$  with a neighborhood of 0 in the normal bundle of  $C_i$ . This bundle splits  $\nu C_i = \nu^+ C_i \oplus \nu^- C_i$  into the positive and negative normal bundles of  $C_i$ . By excision and homotopy equivalence,

$$H_G^*(M_i^+, M_i^-) \cong H_G^*(D_i, S_i)$$

where  $D_i$  and  $S_i$  are the unit disk and sphere bundles, respectively, of  $\nu^- C_i$ . The equivariant Thom isomorphism states that

$$H_G^*(D_i, S_i) \cong H_G^{*-\lambda_i}(C_i)$$

where  $\lambda_i = \dim(\nu^- C_i)$ . Thus the exact sequence (6) is equivalently

$$\dots \rightarrow H_G^{*- \lambda_i}(C_i) \rightarrow H_G^*(M_i^+) \rightarrow H_G^*(M_i^-) \rightarrow H_G^{*+1- \lambda_i}(C_i) \rightarrow \dots \quad (7)$$

LEMMA 2.3. For  $i = 1, \dots, k$ , the sequence (7) splits into the short exact sequence

$$0 \rightarrow H_G^{*- \lambda_i}(C_i) \rightarrow H_G^*(M_i^+) \rightarrow H_G^*(M_i^-) \rightarrow 0. \quad (8)$$

*Proof of Lemma 2.3.* The composition  $H_G^{*- \lambda_i}(C_i) \rightarrow H_G^*(M_i^+) \rightarrow H_G^*(C_i)$ , where the second map is induced by inclusion, restricts to the composition  $H_G^{*- \lambda_i}(C_i) \cong H_G^*(D_i, S_i) \rightarrow H_G^*(D_i) \cong H_G^*(C_i)$ . This latter composition is multiplication by the equivariant Euler class of the negative normal bundle of  $C_i$ . Atiyah and Bott show in [AB1] that, in the case that there is an  $S^1 \subset G$  which is acting on  $\nu^- C_i$  and fixing  $C_i$ , this class is not a zero-divisor. It follows that  $H_G^{*- \lambda_i}(C_i) \rightarrow H_G^*(M_i^+)$  must be an injection. Furthermore, by the exactness of sequence (7), the image of  $H_G^*(M_i^-) \rightarrow H_G^{*+1- \lambda_i}(C_i)$  is the kernel of  $H_G^{*+1- \lambda_i}(C_i) \rightarrow H_G^{*+1}(M_i^+)$ , which is 0 by injectivity. Thus there is a surjection  $H_G^*(M_i^+) \rightarrow H_G^*(M_i^-)$ , showing that the sequence splits.  $\square$

An equivariant cohomology class on  $C_0$  extends to one on  $M_0^+$ . By the homotopy equivalence (5), a class on  $M_i^+$  extends to one on  $M_{i+1}^-$ . Surjectivity in (8) implies that a class on  $M_{i+1}^-$  extends to one on  $M_{i+1}^+$ . Thus by induction a class on  $C_0$  extends to a class in  $H_G^*(M)$ .  $\square$

One may ask the question of how unique these extensions are. By the injection (2), a class is distinguished by its restriction to the fixed point set. As these fixed point sets are critical sets for Morse–Bott functions obtained from components of the moment map, we exploit relationships among the critical sets.

Let  $grad f$  be the gradient of  $f$  with respect to a compatible Riemannian metric. For any critical subset  $C$ , there is a cell-bundle of points  $x \in M$  which converge to  $C$  under the flow of  $-grad f$  (or  $grad f$ ), called the *stable manifold* (or *unstable manifold*) of  $C$ . Furthermore, every point in  $M$  converges to some  $C$  under this flow. Thus, for any  $x \in M$ , there is a (nonunique) sequence of critical sets  $C_{i_1}, C_{i_2}, \dots, C_{i_m}$  such that  $x$  converges to  $C_{i_1}$ , and there are points in the unstable manifold of  $C_{i_j}$  which converge to  $C_{i_{j+1}}$  for every  $j \geq 1$ . Define the *extended stable manifold* of a critical set  $C$  to be the set of points  $x \in M$  whose flows along  $-grad f$  have an associated sequence including  $C$ . In the case that  $M$  is a coadjoint orbit of a semi-simple Lie group, the extended stable manifold of a critical point  $p$  is just the closure of the stable manifold out of  $p$ ; they are called (permuted) Schubert varieties.

LEMMA 2.4. *Suppose  $\alpha \in H_G^*(M)$  restricts to 0 on all  $C_i$  such that  $i < j$ . Then  $\alpha|_{C_j}$  is some multiple of  $e(\nu^-(C_j))$ , the equivariant Euler class of the negative normal bundle of  $C_j$ .*

*Proof.* By (8), a class  $\alpha$  such that  $\alpha|_{M_j^-} = 0$  is in the image of  $H_G^{*-\lambda_j}(C_j) \rightarrow H_G^*(M_j^+)$ . As the map  $H_G^{*-\lambda_j}(C_j) \rightarrow H_G^*(M_j^+) \rightarrow H_G^*(C_j)$  (where the latter map is the restriction to the critical set) is multiplication by  $e(\nu^-C_j)$ ,  $\alpha|_{C_j}$  must be a multiple of this class.  $\square$

LEMMA 2.5. *For every connected component  $C$  of the critical set of  $f$ , there is a class  $\alpha$  with the following restriction properties:*

1.  $\alpha|_{C_i} = 0$  if  $C_i$  is not in the equivariant extended stable manifold of  $C$ , and
2.  $\alpha|_C = e(\nu^-C)$  where  $e(\nu^-C)$  is the equivariant Euler class of the negative normal bundle (defined by  $f$ ) of  $C$ .

*Proof.* Let  $j$  be the index such that  $C_j = C$ . Note that  $C_0, \dots, C_{j-1}$  are not in the extended stable manifold of  $C_j$ , as  $i < j$  implies  $f(C_i) < f(C_j)$ . Using the short exact sequence (8) we extend the class 0 on  $C_0$  to  $\alpha$  on  $M_j^-$  such that  $\alpha$  restricts to 0 on  $C_1, \dots, C_{j-1}$ . As  $1 \in H_G^{*-\lambda_j}(C_j)$  and the composition  $H_G^{*-\lambda_j}(C_j) \rightarrow H_G^*(M_j^+) \rightarrow H_G^*(C_j)$  is multiplication by  $e(\nu^-C_j)$ , we may extend  $\alpha$  to  $M_j^+$  such that its restriction to  $C_j$  is  $e(\nu^-C_j)$ .

Now suppose  $i > j$  but  $C_i$  is not in the extended stable manifold from  $C_j$ . Since points in the unstable manifold of  $C_i$  flow into points in  $M_j^-$ , by Lemma 2.2 there is a CW-complex  $K$  which is  $G$ -homotopic to  $M_j^-$  with a  $\lambda_{C_i}$ -cell bundle over  $C_i$  attached.  $K$  has critical sets  $C_1, \dots, C_{j-1}, C_i$ . By Lemma 2.4  $\alpha|_K$  further restricts to  $m_i e(\nu^-C_i)$  on  $C_i$  where  $m_i \in H_G^*(C_i)$ . Let  $\beta_i \in H_G^*(M)$  be such that  $\beta_i|_{C_l} = 0, l < i$  and  $\beta_i|_{C_i} = m_i e(\nu^-C_i)$ .  $\beta_i$  exists by the first part of this proof, and  $\alpha - \beta_i \in H_G^*(M)$  restricts to 0 on  $C_1, \dots, C_{j-1}, C_i$  and to  $e(\nu^-C_j)$  on  $C_j$ . Let

$$\gamma = \alpha - \sum \beta_i$$

where the sum is over all  $i > j$  such that  $C_i$  is not in the extended stable manifold of  $C_j$ . Then  $\gamma$  has the desired restriction properties.  $\square$

Instead of dealing with CW-complexes, one might consider the closures of the stable manifolds and make the same argument as that above with these varieties. However, the singularities require resolving to be sure there are well-defined classes, restricting one to the complex case only.



### 3 The Equivariant Kirwan Map

In this section we use equivariant Morse theory to prove Theorem 1.2, that the equivariant Kirwan map is surjective. This proof follows very closely Tolman and Weitsman’s rendition of Kirwan’s result (Theorem 1.1) in the case where  $G = S^1$ . The main elements of this proof are found in [TW], modified to allow for a torus action commuting with the  $S^1$  action and for the possibility of non-generic  $T \subset G$ .

*Proof of Theorem 1.2.* Choose  $S^1 \subset G$  and let

$$\Phi_{S^1} : M \rightarrow (\mathfrak{s}^1)^*$$

be a moment map for the  $S^1$  action. We first show that the restriction  $H_G^*(M) \rightarrow H_G^*(\Phi_{S^1}^{-1}(0))$  is surjective.

Let  $\xi \in \mathfrak{g}$  generate the  $S^1$  action. Consider the function  $(\Phi^\xi)^2$  where

$$\Phi^\xi := \langle \Phi, \xi \rangle : M \rightarrow \mathbb{R} .$$

For an appropriate choice of norm on  $(\mathfrak{s}^1)^*$ , we have  $\|\Phi_{S^1}\|^2 = (\Phi^\xi)^2$ . The critical set of  $(\Phi^\xi)^2$  consists of the minimum  $\Phi_{S^1}^{-1}(0)$  and the critical sets of  $\Phi^\xi$ . By the moment map condition (1), the critical points of  $\Phi^\xi$  are the fixed points of the  $S^1$  action generated by  $\xi$ . As Kirwan notes in [K], the function  $(\Phi^\xi)^2$  may not be Morse–Bott function; there may be degenerate critical sets. However, this occurs only at the minimum  $(\Phi^\xi)^{-1}(0)$  where the short exact sequence (8) holds trivially.

As before, without loss of generality we may suppose that  $f = (\Phi^\xi)^2$  separates the critical set. We order them by  $f(C_i) < f(C_j)$  if and only if  $i < j$  and  $C_0 = \Phi_{S^1}^{-1}(0)$ . Let  $M_i^+$  and  $M_i^-$  be as in (3),(4). We use Lemma 2.3 to show that  $H_G^*(M_i^+) \rightarrow H_G^*(\Phi_{S^1}^{-1}(0))$  is surjective for each  $i$ .

We noted that  $H_G^*(M_0^+) \rightarrow H_G^*(C_0)$  is an isomorphism. Now assume that we have a surjection  $H_G^*(M_i^+) \rightarrow H_G^*(C_0)$  for all  $i \leq k - 1$ . By the short exact sequence (8), there is a surjection  $H_G^*(M_k^+) \rightarrow H_G^*(M_k^-)$ . But  $H_G^*(M_k^-) \cong H_G^*(M_{k-1}^+)$  by the homotopy equivalence (5) and the latter ring surjects onto  $H_G^*(C_0)$  by assumption. Thus  $H_G^*(M_k^+) \rightarrow H_G^*(C_0)$  is a surjection. By induction there is a surjection  $H_G^*(M) \rightarrow H_G^*(\Phi_{S^1}^{-1}(0))$ . As 0 is a regular value of  $\Phi_{S^1}$ ,  $H_G^*(\Phi_{S^1}^{-1}(0)) \cong H_{G/S^1}^*(M//S^1)$  and thus

$$H_G^*(M) \rightarrow H_{G/S^1}^*(M//S^1) \tag{9}$$

is a surjection.

There is a residual Hamiltonian  $G/S^1$  action on  $M//S^1$ , which allows us to apply this technique inductively. By reduction in stages, for any commuting subgroups  $H_1$  and  $H_2$  of  $G$ ,  $(M//H_1)//H_2 = M//(H_1 \times H_2)$ .

For any  $T \subset G$ , choose a splitting  $T = S^1 \times \cdots \times S^1$ . Successively apply the surjection (9) to obtain a sequence of surjections

$$H_G^*(M) \rightarrow H_{G/S^1}^*(M//S^1) \rightarrow H_{G/(S^1 \times S^1)}^*(M//(S^1 \times S^1)) \rightarrow \cdots \rightarrow H_{G/T}^*(M//T). \quad \square$$

### 4 The Kernel of the Equivariant Kirwan Map

Here we prove Theorem 1.5, that  $\ker \kappa_T = K_G^{\mathfrak{t}}(\mu)$ , where  $\kappa_T : H_G^*(M) \rightarrow H_{G/T}^*(M//T(\mu))$  is the equivariant Kirwan map. We first show that  $K_G^{\mathfrak{t}}(\mu) \subset \ker \kappa_T$  by directly restricting classes in  $K_G^{\mathfrak{t}}(\mu)$  to  $\Phi_T^{-1}(\mu)$ . We then show the ideals are equal by a dimension count.

*Proof of Theorem 1.5.* Let  $i : \mathfrak{t} \hookrightarrow \mathfrak{g}$  be the inclusion map of Lie algebras and  $\pi : \mathfrak{g}^* \rightarrow \mathfrak{t}^*$  the induced projection. Denote by  $\langle \cdot, \cdot \rangle_G$  and  $\langle \cdot, \cdot \rangle_T$  the natural pairings between  $\mathfrak{g}^*$  and  $\mathfrak{g}$  and between  $\mathfrak{t}^*$  and  $\mathfrak{t}$ , respectively. Then if  $\Phi_G : M \rightarrow \mathfrak{g}^*$  is a moment map for the  $G$  action,  $\Phi_T = \pi \circ \Phi_G$  is a moment map for the restricted  $T$  action. Choose  $\mu \in \mathfrak{t}^*$  a regular value of  $\Phi_T$ . Let  $\alpha \in K_G^\xi$  for some  $\xi \in \mathfrak{t}$ . By definition  $\alpha|_C = 0$  for every connected component  $C$  of  $M^G$  such that  $\Phi_G^{i(\xi)}(C) > \langle \mu, \xi \rangle_T$ . Let

$$M_\xi^+(\mu) = \{m \in M \mid \Phi_G^{i(\xi)}(m) > \langle \mu, \xi \rangle_T\}.$$

$M_\xi^+(\mu)$  is a maximal dimension open  $G$ -invariant submanifold of  $M$ . Thus the restriction of the injection  $H_G^*(M) \rightarrow H_T^*(M^G)$  to  $M_\xi^+(\mu)$  is an injection into the cohomology of those components  $C$  of  $M^G$  which lie in  $M_\xi^+(\mu)$ . Thus  $\alpha|_C = 0$  for all  $C \subset M_\xi^+(\mu)^G$  implies  $\alpha|_{M_\xi^+(\mu)} = 0$ . In particular

$$\alpha|_{\{m \in M \mid \langle \Phi_G(m), i(\xi) \rangle_G = \langle \mu, \xi \rangle_T\}} = 0. \tag{10}$$

But

$$\begin{aligned} \langle \Phi_G(m), i(\xi) \rangle_G &= \langle \pi \circ \Phi_G(m), \xi \rangle_T \\ &= \langle \Phi_T(m), \xi \rangle_T. \end{aligned}$$

Therefore,

$$\begin{aligned} &\{m \in M \mid \langle \Phi_G(m), i(\xi) \rangle_G = \langle \mu, \xi \rangle_T\} \\ &= \{m \in M \mid \langle \Phi_T(m), \xi \rangle_T = \langle \mu, \xi \rangle_T\} \\ &\supseteq \Phi_T^{-1}(\mu). \end{aligned}$$

Thus  $\alpha|_{\Phi_T^{-1}(\mu)} = 0$ , or equivalently  $\alpha \in \ker \kappa_T$ . It follows that any class in the ideal generated by  $K_G^{\mathfrak{t}} = \cup_{\xi \in \mathfrak{t}} K_G^\xi$  lies in  $\ker \kappa_T$ .

To show that the inclusion  $\langle K_G^{\mathfrak{t}} \rangle \subseteq \ker \kappa_T$  is an equality, we prove that

$$\dim \langle K_G^{\mathfrak{t}} \rangle = \dim \ker \kappa_T = \dim (H_{G/T}^*(pt) \otimes \langle K_T^{\mathfrak{t}} \rangle) \tag{11}$$

as graded ideals. We prove this in the case  $\mu = 0$ , as the more general case is identical but notationally more cumbersome.

As  $M$  is a Hamiltonian  $G$ -space and  $M//T$  is a Hamiltonian  $G/T$ -space, they are both equivariantly formal with respect to their group actions. This implies that as graded vector spaces  $H_G^*(M) = H_{G/T}^*(pt) \otimes H_T^*(M)$  and  $H_{G/T}^*(M//T) = H_{G/T}^*(pt) \otimes H^*(M//T)$ . Recall that  $\kappa_T : H_G^*(M) \rightarrow H_{G/T}^*(M//T)$  and  $\kappa : H_T^*(M) \rightarrow H^*(M//T)$  are surjective (Theorems 1.2 and 1.1). Thus there is a graded equality

$$\begin{aligned} \dim \ker \kappa_T &= \dim H_G^*(M) - \dim H_{G/T}^*(M//T) \\ &= \dim (H_{G/T}^*(pt) \otimes H_T^*(M)) - \dim (H_{G/T}^*(pt) \otimes H^*(M//T)) \\ &= \dim (H_{G/T}^*(pt) \otimes \ker(\kappa : H_T^*(M) \rightarrow H^*(M//T))) \\ &= \dim (H_{G/T}^*(pt) \otimes \langle K_T^{\mathfrak{t}} \rangle). \end{aligned}$$

where the last equality follows from Theorem 1.4.

We now show that  $\langle K_G^{\mathfrak{t}} \rangle$  has the same dimension. In degree  $k$  for  $\xi \in \mathfrak{t}$ ,

$$\begin{aligned} \dim \langle K_G^{\xi} \rangle^k &= \dim \{ \alpha \in H_G^k(M) \mid \alpha|_{C_i} = 0 \ \forall i < j, \\ &\quad \text{for any } j \text{ such that } \Phi_G^{\xi}(C_j) > 0 \} \end{aligned}$$

where  $i < j$  if and only if  $\Phi_G^{\xi}(C_i) < \Phi_G^{\xi}(C_j)$ . Let  $F : H_G^*(M) \rightarrow H_T^*(M)$  be the surjective map which forgets the  $G/T$  action. Because  $M^T = M^G$  for generic  $T$ , if  $\alpha \in H_G^k(M)$  has the property that  $\alpha|_{C_i} = 0 \ \forall i < j$ , then  $F(\alpha) \in H_T^k(M)$  has the property that  $F(\alpha)|_{C_i} = 0 \ \forall i < j$ . Furthermore, for  $\xi \in \mathfrak{t}$ ,  $\Phi_T^{\xi} = \Phi_G^{\xi}$  so that such classes are precisely those in  $\langle K_T^{\xi} \rangle$ . As  $\ker F = H_{G/T}^*(pt)$  in all but degree 0, we conclude that

$$\begin{aligned} \dim \{ \alpha \in H_G^k(M) \mid \alpha|_{C_i} = 0 \ \forall i < j, \text{ for any } j \text{ such that } \Phi_G^{\xi}(C_j) > 0 \} \\ &= \dim \left\langle \left\{ \beta \in \sum_{l+m=k} H_{G/T}^l(pt) \otimes H_T^m(M) \mid \beta|_{C_i} = 0 \ \forall i < j, \right. \right. \\ &\quad \left. \left. \text{for any } j \text{ such that } \Phi_G^{\xi}(C_j) > 0 \right\} \right\rangle \\ &= \dim^k (H_{G/T}^*(pt) \otimes \langle K_T^{\xi} \rangle) \quad \square \end{aligned}$$

### 5 The Walls of the Moment Polytope

In this section, we prove an important refinement of Theorem 1.5. It states that the collection of  $\langle K_G^\xi(\mu) \rangle$  for a small number of  $\xi \in \mathfrak{g}$  are sufficient to generate the kernel of the map  $H_G^*(M) \rightarrow H^*(M//G(\mu))$ . In particular, one can consider only such  $\xi$  which are perpendicular to codimension-1 walls of the moment polytope.

We illustrate the main theorem of this section with an example. Let  $M$  be a generic 6-dimensional coadjoint orbit of  $SU(3)$ . The maximal torus  $G = T^2$  of  $SU(3)$  acts on this orbit in a Hamiltonian fashion. The image of the moment map is a hexagon, and the codimension-1 walls of the moment polytope are shown in Figure 1(a).

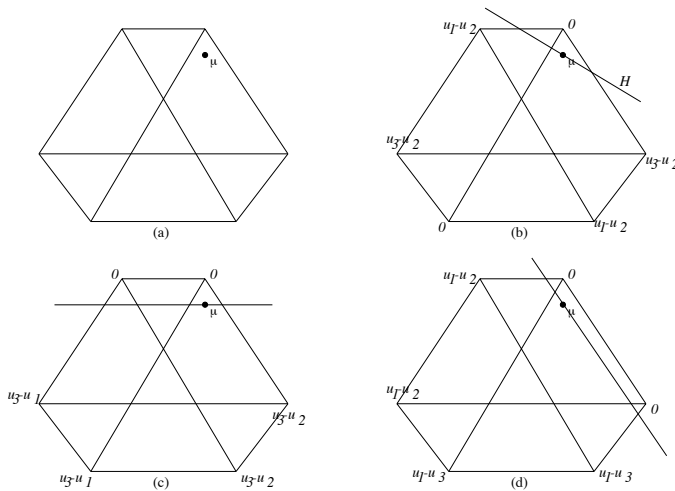


Figure 1: (a) The image of a generic  $SU(3)$  coadjoint orbit under the moment map; (b) A class  $\alpha$  that is 0 to one side of a codimension-1 hyperplane through  $\mu$ ; (c)-(d) Two classes which restrict to 0 on one side of a hyperplane parallel to a wall of the moment polytope. Their sum is  $\alpha$ .

Consider the reduction at the point  $\mu$  indicated. According to Theorem 1.4, a generating set for the kernel of the map  $H_G^*(M) \rightarrow H^*(M//G(\mu))$  would include classes which restrict to 0 on fixed points whose images under the moment map (indicated by vertices) lie to one side of any hyperplane through  $\mu$ . In particular, the class  $\alpha$  whose restrictions to the fixed points is indicated by Figure 1(b) would be a generator of the kernel because it is 0 to one side of the hyperplane  $H$ . Theorem 5.1 states that such a class is redundant; it will in fact be generated by classes which are 0 to one side

of a hyperplane through  $\mu$  parallel to a wall of the moment polytope. In Figures 1(c) and 1(d) we see two classes which are in the kernel according to Theorem 5.1 and whose sum is  $\alpha$ .

More generally, let  $C$  be a connected fixed point component of the  $G$ -action on  $M$ , and  $\xi \in \mathfrak{g}$  generic so that  $C$  is a critical manifold for  $\Phi^\xi$ . Let  $X^\xi$  be the extended stable manifold of  $C$  under  $\Phi^\xi$ . The image  $\Phi(X^\xi)$  of  $X^\xi$  under the moment map is not *a priori* convex. Suppose  $\dim \Phi(X^\xi) = \dim \mathfrak{g}^* = d$ . Consider the collection of codimension-1 walls of  $\Phi(M)$  that lie in  $\Phi(X^\xi)$ . These are the images of  $S^1$ -fixed point sets in  $X^\xi$ . At least one of these walls, translated to pass through  $C$ , has the property that  $\Phi(X^\xi)$  lies entirely to one side. If  $\Phi(X^\xi)$  is not maximal dimension, then every codimension-1 wall containing  $\Phi(X^\xi)$  has this property.

We are now ready to state and prove in what sense the walls of the moment polytope are sufficient information for calculating the kernel of  $\kappa$ .

**Theorem 5.1.** *Let  $\Xi_\mu^\perp$  consist of hyperplanes through  $\mu$  and parallel to codimension-1 walls of the moment polytope. Let  $K \subset H_G^*(M)$  be the ideal generated by classes  $\alpha \in H_G^*(M)$  which restrict to 0 on all connected components  $C$  of  $M^G$  whose images under  $\Phi$  lie to one side of some  $H \in \Xi_\mu^\perp$ . Then  $K = \ker \kappa$ , where  $\kappa : H_G^*(M) \rightarrow H^*(M//G(\mu))$  is the Kirwan map.*

*Proof.* Choose any  $\alpha \in H_G^*(M)$  where  $\alpha|_{\Phi^{-1}(\mu)} = 0$ . By Theorem 1.4,  $\alpha \in \sum_{\xi \in \mathfrak{g}} K_G^\xi(\mu)$ . We want to show that  $\alpha$  can be written as a linear combination of elements in  $K_G^\eta(\mu)$  where  $\eta \in \Xi$  are the annihilators of the hyperplanes through 0 and parallel to codimension-1 walls of the moment polytope.

Without loss of generality, assume  $\alpha \in K_G^\xi(\mu)$  for some  $\xi$ , where  $K_G^\xi(\mu)$  are classes restricting to 0 on fixed points whose image under  $\Phi$  lies to one side of  $\xi^\perp + \mu$ . Order the connected components of the critical sets  $C_1, \dots, C_l$  so that  $i < j$  if and only if  $\Phi^\xi(C_i) < \Phi^\xi(C_j)$ . We prove that  $\alpha$  can be expressed as a sum of elements in  $K_G^\eta(\mu), \eta \in \Xi$ , by induction on the index of the critical sets. Let  $C_{i_1}$  be the first critical set such that  $\alpha|_{C_{i_1}} \neq 0$ . Then  $\alpha|_{C_{i_1}}$  is some multiple  $m_{i_1}$  of  $e(\nu_{\Phi^\xi}^- C_{i_1})$ . Let  $X_1^\xi$  be the extended stable manifold of  $C_{i_1}$  and let  $\alpha_{i_1}$  be any class satisfying the properties of Lemma 2.5. In particular,  $\alpha_{i_1} \in K_G^{\eta_{i_1}}(\mu) \cup K_G^{-\eta_{i_1}}(\mu)$  where  $\eta_{i_1}$  is perpendicular to a codimension-1 wall  $H$  of  $\Phi(M)$  such that  $\Phi(X_1^\xi)$  lies to one side of  $H$  shifted to pass through  $\mu$ . Then  $\alpha - m_{i_1}\alpha_{i_1}$  is a class which restricts to 0 on  $C_1, \dots, C_{i_1}$ . Now suppose that  $\alpha - \sum_{k=1}^l \alpha_{i_k}$  restricts to 0 on  $C_1, \dots, C_{i_l}$ . Let  $C_{i_{l+1}}$  be the first critical set on which this class is non-zero.

Use Lemma 2.5 to find a class  $\alpha_{i_{l+1}}$  supported on  $X_{l+1}^\xi$ . Then  $\alpha - \sum_{k=1}^{l+1} \alpha_{i_k}$  is 0 on  $C_1, \dots, C_{i_{l+1}}$ . In this manner, we express  $\alpha = \sum_i \alpha_i$ , where each  $\alpha_i \in K_G^\eta(\mu)$  for some choice of  $\eta$  perpendicular to a codimension-1 wall of the polytope.  $\square$

### 6 Application to the Product of Symplectic Manifolds

Let  $M = X_1 \times \dots \times X_k$  be the product of symplectic manifolds  $X_i$ , each with a Hamiltonian  $T$  action. Theorem 5.1 allows us to say a lot about the reduction of  $M$  by the diagonal torus action. We note that the diagonal torus  $T_\Delta$  is a subtorus of the product  $G = T \times \dots \times T$  acting on  $M$ . If the torus is just one-dimensional, Theorem 5.1 is not more useful than the original formulation of Theorem 1.4. It is when  $\dim T \geq 2$  that one can significantly reduce the number of vectors needed to generate the kernel of  $\kappa : H_{T_\Delta}^*(M) \rightarrow H^*(M//T_\Delta)$ . We calculate the cohomology of the reduced space for a product of two copies of  $\mathbb{C}P^2$ s, where we have quotiented by the diagonal  $T_\Delta^2$  action.

Let  $X_1 = X_2 = \mathbb{C}P^2$  and  $T$  act on  $X_i$  by  $(\theta_1, \theta_2) \cdot [z_0 : z_1 : z_2] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2]$ . By the moment map condition (1), if we choose the symplectic forms  $\omega$  on  $X_1$  and  $k\omega$ ,  $k \in \mathbb{R}$  on  $X_2$ , the image for the moment map for the  $T$  action on  $X_2$  is that for the action of  $T$  on  $X_1$  diluted by  $k$ . The image of the moment map for the diagonal  $T$  action on  $X_1 \times X_2$  is the sum of the moment maps for each component. Choose  $k > 2$  and let  $\Phi : X_1 \times X_2 \rightarrow \mathfrak{t}_\Delta^*$  be a moment map for the diagonal action. Then  $\Phi(X_1 \times X_2)$  with its walls is pictured in Figure 2.

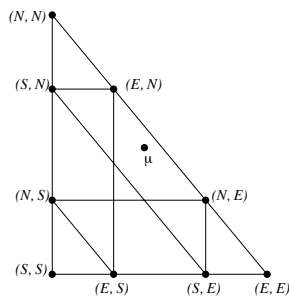


Figure 2: The image of the moment map for  $T_\Delta^2$  acting on  $\mathbb{C}P^2 \times \mathbb{C}P^2$ .

The image of fixed points are labeled and indicated by vertex dots, and the image of fixed point sets of codimension-1 tori are indicated by

line segments, which in this case form codimension-1 walls of the moment polytope. We find the cohomology of the symplectic reduction  $(X_1 \times X_2)//T_\Delta(\mu)$  where  $\mu$  is the point indicated in the figure by finding the kernel of  $\kappa : H_{T_\Delta}^*(X_1 \times X_2) \rightarrow H^*((X_1 \times X_2)//T_\Delta(\mu))$  using Theorem 5.1. We note that there are three distinct hyperplanes through  $\mu$  parallel to walls of the moment polytope: horizontal, vertical, and diagonal with slope -1.

Consider first  $H_{T_\Delta}^*(X_1 \times X_2)$ . By the equivariant Künneth theorem,

$$H_{T_\Delta}^*(X_1 \times X_2) = H_T^*(X_1) \otimes_{H_T^*(pt)} H_T^*(X_2), \tag{12}$$

where  $T_\Delta \subset G = T \times T$  is the diagonal torus.  $H_T^*(\mathbb{C}P^2)$  is generated in degree 2, by characteristic classes inherited from the module structure  $H_T^2(pt) \rightarrow H_T^2(\mathbb{C}P^2)$ , and the equivariant symplectic form on  $\mathbb{C}P^2$ . We use the chosen basis for  $\mathfrak{t}^* \cong H_T^2(pt)$  and denote the characteristic classes by  $u_1$  and  $u_2$ . These classes restrict to themselves on each fixed point of  $T$  on  $\mathbb{C}P^2$ . Let  $x$  be (the multiple of) the equivariant symplectic class given by the restrictions indicated in Figure 3.

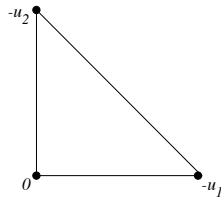


Figure 3: The restriction of the class  $x$  to the fixed point set of  $T$  acting on  $\mathbb{C}P^2$ .

Similarly,  $u_1$  and  $u_2$  are degree 2 equivariant classes on  $\mathbb{C}P^2 \times \mathbb{C}P^2$  which restrict to themselves at every fixed point of the  $T_\Delta$  action on the product. By (12) the classes  $u_1, u_2, x \otimes 1$  and  $1 \otimes x$  generate  $H_{T_\Delta}^*(\mathbb{C}P^2 \times \mathbb{C}P^2)$ . Here  $u_i = u_i \otimes 1 = 1 \otimes u_i$ .

Note that a fixed point of  $\mathbb{C}P^2 \times \mathbb{C}P^2$  is a pair  $(p, q) \in ((\mathbb{C}P^2)^T, (\mathbb{C}P^2)^T)$ . The restriction of a class on the product space to a fixed point is

$$(a \otimes b)|_{(p,q)} = a|_p \otimes b|_q \tag{13}$$

where  $a, b \in H_T^*(\mathbb{C}P^2)$ . The algebraic structure of these classes is inherited by multiplication on each fixed point.

The Betti numbers for  $H_{T_\Delta}^*(\mathbb{C}P^2 \times \mathbb{C}P^2)$  are easy to compute. As graded vector spaces,  $H_{T_\Delta}^*(\mathbb{C}P^2 \times \mathbb{C}P^2) = H_T^*(pt) \otimes H^*(\mathbb{C}P^2 \times \mathbb{C}P^2)$ . It follows that the equivariant Poincaré polynomial for  $\mathbb{C}P^2 \times \mathbb{C}P^2$  is  $(1 + t^2 + t^4 + \dots)^2(1 + t^2 + t^4)^2 = 1 + 4t^2 + 10t^4 + \dots$ . We noted above

that  $u_1, u_2, 1 \otimes x$ , and  $x \otimes 1$  are four linearly independent degree 2 classes. A choice of 10 linearly independent degree 4 classes is  $u_1^2, u_2^2, u_1 u_2, x \otimes u_1, x \otimes u_2, u_1 \otimes x, u_2 \otimes x, x \otimes x, 1 \otimes x^2$ , and  $x^2 \otimes 1$ . The restrictions of these classes to the fixed point set  $(\mathbb{C}P^2 \times \mathbb{C}P^2)^{T_\Delta}$  is determined by the formula (13). As an example, we compute the restriction of  $x \otimes x$  to the fixed point set. See Figure 4.

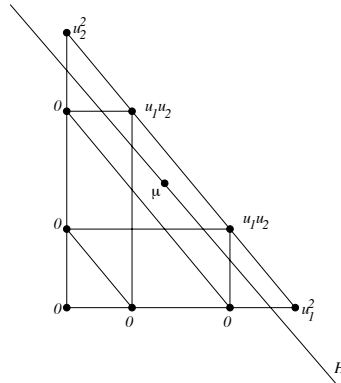


Figure 4: The restriction of the class  $x \otimes x$  to the fixed point set of the diagonal  $T$  acting on  $\mathbb{C}P^2 \times \mathbb{C}P^2$ . Notice that this class restricts to 0 on fixed points whose image under  $\Phi$  lie on one side of the indicated hyperplane through  $\mu$ .

We have left to find the kernel of  $\kappa : H_{T_\Delta}^*(\mathbb{C}P^2 \times \mathbb{C}P^2) \rightarrow H^*((\mathbb{C}P^2 \times \mathbb{C}P^2)//T_\Delta(\mu))$ . The reduced space  $(\mathbb{C}P^2 \times \mathbb{C}P^2)//T_\Delta(\mu)$  has (real) dimension

$$\dim(\mathbb{C}P^2 \times \mathbb{C}P^2) - 2 \dim T_\Delta = 8 - 4 = 4.$$

Thus we expect all classes of degree 6 or higher to be in the kernel of  $\kappa : H_{T_\Delta}^*(\mathbb{C}P^2 \times \mathbb{C}P^2) \rightarrow H^*((\mathbb{C}P^2 \times \mathbb{C}P^2)//T_\Delta(\mu))$ . In degree 2, we can easily see that no linear combination of the four classes above will restrict to 0 on one side of any of the three hyperplanes parallel to walls of the moment polytope. By Theorem 5.1, there are no degree 2 classes in  $\ker \kappa$ . In degree 4, however, we expect nine (of ten) linearly independent classes to be in the kernel, as the image of  $\kappa$  is (a multiple of) the volume form on the reduction. The reader can verify using (13) that the following classes restrict to 0 on one side of one of the three hyperplanes through  $\mu$  parallel to a wall of the moment polytope:  $x \otimes x, u_1 u_2 + u_1 \otimes x, u_1^2 + u_1 \otimes x, u_1^2 - 1 \otimes x^2, u_2^2 - 1 \otimes x^2, u_2^2 - u_2 \otimes x, u_1 \otimes x - x \otimes u_1 + u_2 \otimes x - x \otimes u_2 + x^2 \otimes 1 - 1 \otimes x^2, 1 \otimes x^2 + u_1 \otimes x$ , and  $x \otimes u_2 + x \otimes u_1 + x^2 \otimes 1 + u_1 u_2$ . They are all linearly independent, which



can be verified rather tediously. As an example, one can see in Figure 4 that  $x \otimes x$  restricts to 0 on one side of the diagonal hyperplane through  $\mu$ .

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