MATH 351
Solutions #6

1. Suppose
$$f(x) = \begin{cases} c(1-x^2) & \text{if } -2 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$
Is there a value of c for which f is a probability density function? Why or why not?

   Solution. This cannot be a probability density function. If $c = 0$, then it does not integrate 1. For any $c \neq 0$, there is an interval in $-2 \le x \le 2$ over which the integral is negative, and therefore does not represent a probability over this interval

2. Suppose that
$$f(x) = \begin{cases} c(3x-x^2) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$
is the probability density function of the random variable X.

   (a) Find c.

   Solution. We integrate and set this equal to 1. This gives us $c = 3/10$.

   (b) Find $P\{-1 \le x \le 1\}$.

   Solution.
$$\begin{aligned} P\{-1 \le x \le 1\} &= \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 0\,dx + \int_{0}^{1} \frac{3}{10}(3x-x^2)\,dx \\ &= \frac{3}{10}(\frac{3x^2}{2} - \frac{x^3}{3})\Big|_0^1 = \frac{7}{20} \end{aligned}$$

   (c) Find $E[X]$.

   Solution.
$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^2 x\frac{3}{10}(3x-x^2)\,dx \\ &= \frac{3}{10}\int_0^2 (3x^2 - x^3)\,dx = \frac{3}{10}x^3 - \frac{x^4}{4}\Big|_0^2 = \frac{6}{5} \end{aligned}$$

3. Suppose X is a random variable with probability density function

$$f(x) = \begin{cases} cxe^{-x} & \text{if } x \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c.

Solution. We have to solve for c:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{2}^{\infty} cxe^{-x} dx = 1.$$

We use integration by parts, letting $u = x$, and $dv = e^{-x}\,dx$, making $du = dx$ and $v = -e^{-x}$ to obtain

$$c\left((x)(-e^{-x})\Big|_2^\infty - \int_2^\infty (-e^{-x})\,dx \right).$$

We use L'Hopital's rule to evaluate the limit $\lim_{x\to\infty} -\frac{x}{e^{-x}} = \lim_{x\to\infty} \frac{1}{e^{-x}} = 0$. Thus

$$c\left((x)(-e^{-x})\Big|_2^\infty + \int_2^\infty (e^{-x})\,dx \right) = 2e^{-2} - e^{-x}\Big|_2^\infty = c(2e^{-2}+e^{-2}).$$

Therefore, $c = \frac{e^2}{3}$.

(b) Find $E[X]$.

Solution. We use integration by parts:

$$\begin{aligned} E[X] &= \int_2^\infty \frac{e^2}{3} x^2 e^{-x}\,dx \qquad u = \frac{e^2}{3}x^2, dv = e^{-x}\,dx, du = \frac{2}{3}e^2 x\,dx, v = -e^{-x} \\ &= -\frac{e^2}{3}x^2 e^{-x}\Big|_2^\infty + \int_2^\infty \frac{2}{3}e^2 x e^{-x}\,dx \\ &= \frac{e^2}{3} \cdot 4e^{-2} + \frac{2e^2}{3}(-xe^{-x}\Big|_2^\infty + \int_2^\infty e^{-x}\,dx) \\ &= \frac{4}{3} + \frac{4}{3} - \frac{2e^2}{3}e^{-x}\Big|_2^\infty = \frac{10}{3} \end{aligned}$$

(Watch your signs! I didn't write out each sign step).

4. Describe, in words, whether you think the likelihood of a hurricane during a given period of time is best described in terms of a Poisson distribution or a uniform distribution. Give reasons for your answer.

   Solution. This is decidedly *not* a uniform distribution. The reason is *not* that hurricanes have seasonal changes in probability – though this is true, the question suggests that a given period of time might be shorter – like a particular month or even a week or a day. Here is a better reason. Suppose it was modeled by a uniform distribution. No matter what probability you might have over a period of time, the integral of the probability is one over all time. But we have no guarantees that a hurricane will actually happen over this period of time. There is, in fact, for any given period of time, a probability that there are no hurricanes. This is a major drawback of the uniform distribution. The other reason Poisson is better is that Poisson is discrete – and so is the outcome if X is the random variable given by, say, the number of hurricanes in a given time period. Generally Poisson is good random variable to model situations in which you want to find the probability a specific event will occur in a fixed time period.

5. Trains leaving Penn Station, New York to New Jersey leave the station every ten minutes. A man arrives at the station at a random time. Let X be the time he will have to wait for the next train to leave.

   (a) What kind of random variable is X?

   Solution. This is a uniform random variable with pdf given by

$$f(x) = \begin{cases} \frac{1}{10} & 0 \leq x \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

   (b) What is $P\{X \geq 4\}$?

   Solution.

$$\int_4^\infty f(x)\,dx = \int_r^1 0\frac{1}{10}\,dx = \frac{3}{5}$$

   (c) Find $E[X]$ and $Var(X)$.

   Solution.

$$E[X] = \int_0^{10} \frac{x}{10}\,dx = \left.\frac{x^2}{20}\right|_0^{10} = 5$$

Recall that $Var(X) = E[X^2] - E[X]^2$. We have $E[X^2] = \int_0^{10} x^2/10\,dx = 100/3$. Thus $Var(X) = \frac{100}{3} - 25 = \frac{25}{3}$.

6. Suppose that X is a uniform random variable on the interval $[-1, 1]$. Find the probability density functions of $X, |X|$, and e^X.

   Since X is uniform on an interval of length 2, the probability density function is given by

$$f_X(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   For $|X|$, we first find its cdf. For $0 \le t \le 1$,

$$F_{|X|} = P(\{|X| \le t\}) = P(\{-t \le X \le t\} = \int_{-t}^{t} \frac{1}{2}\,dx = t$$

   Therefore

$$f_{|X|}(t) = \frac{d}{dt}F_{|X|}(t) = \begin{cases} 1 \text{ if } 0 \le t \le 1 \\ 0 \text{ otherwise.} \end{cases}$$

   Similarly, for $-1 \le lnt \le t$, or, equivalently, $e^{-1} \le t \le e$,

$$F_{e^X}(t) = P(\{e^X \le t\}) = P(\{X \le \ln t\} = \int_{-1}^{\ln t} \frac{1}{2}\,dx = \frac{1 + \ln t}{2}.$$

   The probability density function is therefore

$$f_{e^X}(t) = \frac{d}{dt}F_{e^X}(t) = \begin{cases} \frac{1}{2t} \text{ if } e^{-1} \le t \le t \\ 0 \text{ else.} \end{cases}$$