olutions #8

1. a) $p(x,y) = \begin{cases} \frac{11}{36} \cdot \left(\frac{25}{36}\right)^{x-1} & \text{if } x=y, \quad x,y \text{ positive integers} \\ 0 & \text{otherwise} \end{cases}$

Note that one of the dice is 1 for pairs $(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$, $(2,1)$, ~~(2,1), (2,2)~~, $(3,1), (4,1), (5,1)$ $(6,1)$

i.e. $\frac{11}{36}$ of the rolls.

So $P\{X=n, Y=n\}$ is when one of the 25 other rolls occur (chance $25/36$ for each roll) the first $n-1$ rolls, then one of the 11 rolls with a "1" occurs on n^{th} roll.

b) $p_Y(y) = \begin{cases} \left(\frac{25}{36}\right)^{y-1}\left(\frac{11}{36}\right) & y \text{ positive integer} \\ 0 & \text{otherwise} \end{cases}$

c) $E(X+2Y) = E(X) + 2E(Y)$

Since $X=Y$, we have

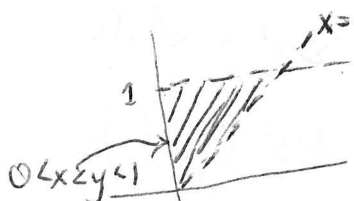
$$E[X+2Y] = \sum_{x=1}^{\infty} 3x \left(\frac{25}{36}\right)^{x-1}\left(\frac{11}{36}\right)$$

The $3x$ comes from $x+2y$ where $x=y$.

$$= 3 \underbrace{\sum_{x=1}^{\infty} x \left(\frac{25}{36}\right)^{x-1}\left(\frac{11}{36}\right)}_{\text{expected value of geom. r.v. with } p=\frac{11}{36} \text{ is } \frac{1}{p}}$$

$$= 3\left(\frac{36}{11}\right) = \frac{72}{11}$$

2. (a)



(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^y 2\,dx\,dy$

$= \int_0^1 2y\,dy = y^2\Big|_0^1 = 1$ ✓

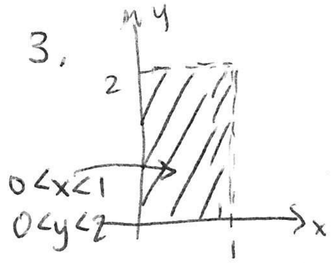
(c) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^1 2\,dy = 2-2x \Rightarrow f_X(x) = \begin{cases} 2-2x & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y 2\,dx = 2y \Rightarrow f_Y(y) = \begin{cases} 2y & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

(d) Note $f(\frac{1}{2}, \frac{1}{4}) = 0$ since $\frac{1}{2} > \frac{1}{4}$. But $f_X(\frac{1}{2}) = 1$ and $f_Y(\frac{1}{4}) = \frac{1}{2}$

$\Rightarrow f(\frac{1}{2}, \frac{1}{4}) \neq f_X(1/2) f_Y(\frac{1}{4}) \Rightarrow$ NOT INDEPENDENT

3.



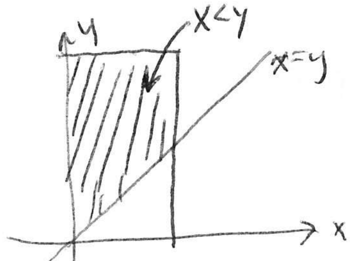
a) $f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 xy\,dy = x\frac{y^2}{2}\Big|_{y=0}^{2} = 2x$ for $x\in(0,1)$

$$\Rightarrow f_X(x)=\begin{cases}2x & 0<x<1\\ 0 & o.w.\end{cases}$$

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^1 xy\,dx = \frac{x^2}{2}y\Big|_0^1=\frac{1}{2}y$$

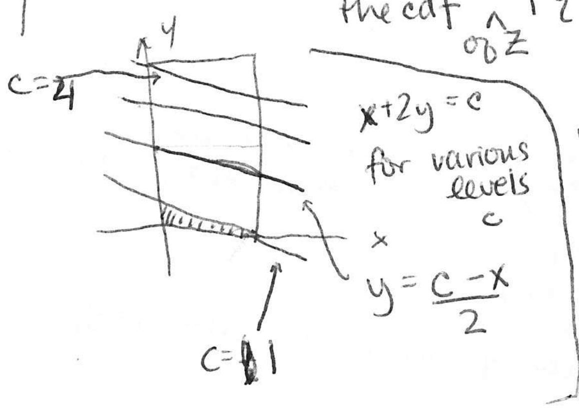
$$\Rightarrow f_Y(y)=\begin{cases}\frac{1}{2}y & 0<y<2\\ 0 & o.w.\end{cases}$$

b) $P\{X<Y\}=\int_0^1\int_x^2 xy\,dy\,dx=\int_0^1 x\frac{y^2}{2}\Big|_x^2\,dx = \int_0^1\left(2x-\frac{x^3}{2}\right)dx$



$$= x^2-\frac{x^4}{8}\Big|_0^1 = \boxed{\frac{7}{8}}$$

c) Let $Z=X+2Y$. We want $f(z)$. Let's find the cdf of Z $P\{Z\le c\} = \int_0^1\int_0^{\frac{c-x}{2}} xy\,dy\,dx$ $\quad 1<c<4$



When $c>1$ and less than 4, the x-integration limits are 0 to 1

$$= \int_0^1 x\frac{y^2}{2}\Big|_{y=0}^{y=\frac{c-x}{2}}\,dx$$

$$= \int_0^1 \frac{x\left(\frac{c-x}{2}\right)^2}{2}\,dx = \frac{1}{8}\int_0^1 x(c-x)^2\,dx$$

$$= \frac{1}{8}\int_0^1 (c^2x-2cx^2+x^3)\,dx$$

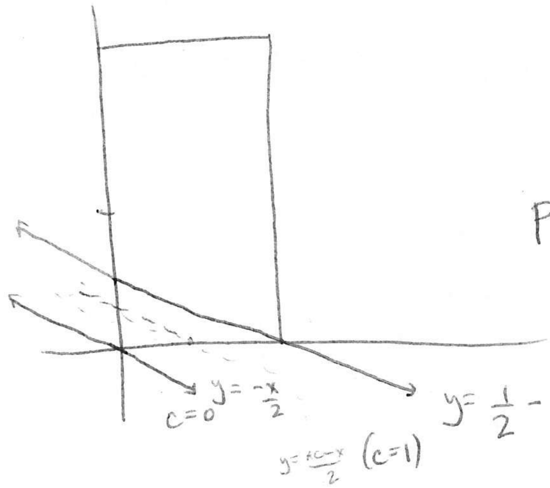
$$= \frac{1}{8}\left(\frac{c^2x^2}{2}-\frac{2cx^3}{3}+\frac{x^4}{4}\Big|_0^1\right)=$$

$$= \frac{1}{16}c^2-\frac{2c}{3}+\frac{1}{4}$$

And the pdf is the derivative, ie.

$$f(z)=\frac{1}{8}z-\frac{2}{3} \quad \text{for } 1<z<4$$

3(c) cont.



Lines for $0<c<1$ intersect the y-axis when $x=c$.

For $0<c<1$,

$$P\{\le c\} = \int_{0}^{x=c}\int_{y=0}^{y=\frac{c-x}{2}} x y \, dy \, dx$$

$$= \frac{1}{8}\int_0^c x(c-x)^2 \, dx$$

$$= \frac{1}{8}\ \frac{c^2x^2}{2} - 2c\frac{x^3}{3} + \frac{x^4}{4}\Big|_0^c$$

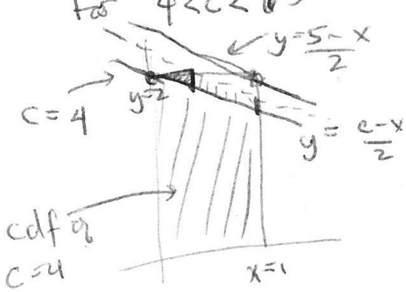
$$= \frac{1}{8}\left(\frac{c^4}{2} - \frac{2}{3}c^4 + \frac{c^4}{4}\right)$$

$$= \frac{1}{8}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)c^4 = \frac{1}{8}\left(\frac{6}{12} - \frac{8}{12} + \frac{3}{12}\right)c^4$$

$$= \frac{1}{8}\left(\frac{1}{12}\right)c^4 = \frac{c^4}{96}$$

The pdf is then $\frac{d}{dc}\frac{c^4}{96} = \frac{1}{24}c^3$ for $0<c<1$.

For $4<c<5$



$$P\{X \le c\} = \underbrace{\frac{1}{16}(4)^2 - \frac{2\cdot(4)}{3} + \frac{1}{4}}_{\text{the cdf of } c=4} + \int_{c-4}\int_{y=2-\frac{x}{2}}^{y=\frac{c-x}{2}} x y \, dx$$

$$+ \int_0^{c-4}\int_{\frac{c-x}{2}}^{2} x y \, dx$$

$y = \frac{c-x}{2}$ intersects $y=2$

when $2 = \frac{c-x}{2}$, $\Rightarrow 4 = c-x$

$\Rightarrow x = c-4$

height of small triangle is

$$\frac{c-x}{2} - \frac{4-x}{2} = \frac{c}{2} - 2$$

I won't bother to calculate this. The pdf is its derivative (for $4<c<5$).

Putting these together, get pdf of $X+2Y$.

4. First, we find the cdf. For $0<t<1$,

$P\{\min(X,Y)<t\}$

$= P\{X<t \;\&\; Y<t\} + P\{X<t \;\&\; Y>t\} + P\{Y<t \;\&\; X>t\}$

(note that these three outcomes are mutually exclusive).

$= P\{X<t\}P\{Y<t\} + P\{X<t\}P\{Y>t\} + P\{Y<t\}P\{X>t\}$

(since X and Y are independent)

$= t \cdot t + t(1-t) + (t)(1-t)$

(Since X and Y are uniformly distributed)

$= t^2 + 2t(1-t) = t^2+2t-2t^2 = 2t-t^2.$

Alternatively: for $0<t<1$,

$P\{\min(X,Y)<t\} = 1 - P\{X>t \;\&\; Y>t\}$

$= 1 - P\{X>t\}P\{Y>t\}$ by independence

$= 1-(1-t)(1-t)$ since they are uniformly distributed

$= 1 - (1-2t+t^2)$

$= 2t-t^2$

For $t<0$ the cdf is 0. For $t>1$, the cdf is 1.

Thus pdf of $\min(X,Y) = \frac{d}{dt}(2t-t^2) = \begin{cases} 2-2t & 0<t<1 \\ 0 & \text{otherwise.} \end{cases}$

$= \frac{d}{dt}(0) = 0$

or $\frac{d}{dt}(1)$

5 (a). $P\{X > Y\} = P\{X - Y > 0\}$

Let $W = X - Y$. Since W is a sum of two normal random variables

$X \sim N(5, 9)$ and $-Y \sim N(+1, 4)$

note the average got negated compared to Y.

note the variance stayed the same compared to $Var(Y)$.

we know $W \sim N(5+1, 9+4)$

$W \sim N(6, 13)$

$$P\{W > 0\} = P\left\{\frac{W-6}{\sqrt{13}} > \frac{-6}{\sqrt{13}}\right\} = P\left\{Z_{std} > \frac{-6}{\sqrt{13}}\right\}$$

$$= P\left\{Z_{std} < \frac{6}{\sqrt{13}}\right\} \approx P\{Z_{std} < 1.664\} \approx .95$$

(b) $P\{X+Y > 3\} = P\{W > 3\} = P\left\{\frac{W-4}{\sqrt{13}} > \frac{3-4}{\sqrt{13}}\right\}$

Let $W = X + Y$

$W \sim N(5-1, 9+4)$

$N(4, 13)$

$$= P\left\{Z_{std} > \frac{-1}{\sqrt{13}}\right\}$$

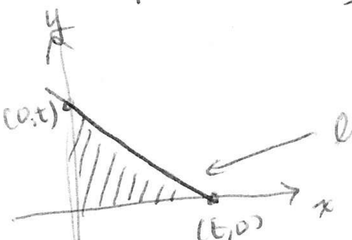
$$= P\left\{Z_{std} < \frac{1}{\sqrt{13}}\right\}$$

$$\approx P\{Z_{std} < .277\} \approx .609$$

6. We first find the cdf. Recall that the pdfs of X and Y take on non-~~neg~~ zero values when $x, y \geq 0$.

$$f_X(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$P\{X+Y \leq t\} =$



$$f(x,y) = f_X(x) f_Y(y) = 2e^{-2x} 2e^{-2y} \quad x,y \geq 0$$

since X, Y independent

$$\int_0^t \int_0^{t-y} 4e^{-2x} e^{-2y} \, dx \, dy = \int_0^t 4e^{-2y} \left(\int_0^{t-y} e^{-2x} dx \right) dy$$

$$= \int_0^t 4e^{-2y} \left(-\tfrac{1}{2}\right)\left(e^{-2(t-y)} - e^0\right) dy$$

$$= \int_0^t \left(-2e^{-2t} + 2e^{-2y}\right) dy = -2te^{-2t} + -e^{-2y} \Big|_0^t$$

$$= -2te^{-2t} - e^{-2t} + 1.$$

The pdf is the derivative:

$$\frac{d}{dt}\left(-2te^{-2t} - e^{-2t} + 1\right) = -2e^{-2t} + 4te^{-2t} + 2e^{-2t} = 4te^{-2t}$$

$$\Rightarrow \quad f_{X+Y}(t) = \begin{cases} 4te^{-2t} & t \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

You can also solve this using convolutions:

Let $Z = X + Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) \, dy$$

$$= \int_0^z 2e^{-2(z-y)} 2e^{-2y} \, dy$$

$$= \int_0^z 4e^{-2z} \, dy = 4ze^{-2z}.$$

$$\Rightarrow \quad f_Z(z) = \begin{cases} 4ze^{-2z} & z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

7. $P\{Z > X+Y\} = \int_0^\infty \int_0^\infty \int_{x+y}^\infty e^{-(x+y+z)}\, dz\, dy\, dx$

keep in mind the pdfs are 0 except for when $x \geq 0, y \geq 0$.

$f_X(x) f_Y(y) f_Z(z) = 1e^{-x} \cdot 1e^{-y} \cdot 1e^{-z}$
$= e^{-(x+y+z)}$
when $x, y, z \geq 0$.

$$= \int_0^\infty \int_0^\infty -e^{-(x+y+z)} \Big|_{z=x+y}^{\infty} dy\, dx = \int_0^\infty \int_0^\infty e^{-2(x+y)}\, dy\, dx$$

$$= \int_0^\infty \frac{1}{2} e^{-2x}\, dx = \frac{1}{4}.$$

Multiply by 3 for $X > Y + Z$, $Y > X + Z$. Answer $= 3/4$.