Math 678.
Lecture 8.

$u_t - \Delta u = 0$ — Heat eqn

$u_t - \Delta u = f$ — nonhom. Heat eqn

1D motivation :
$$\begin{cases} D\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} \\ u(x,0) = \cancel{u_0(x)}\ 0 \quad (t=0) \\ u(x\to\infty) = 0 \\ u(x=0) = u_0 \end{cases}$$

Dimensional analysis : $[u] = \dfrac{M}{L^3}$ - density

$$[u_t] = \frac{[u]}{T}, \quad [u_{xx}] = \frac{[u]}{L^2}$$

$$\frac{[D]\cdot M}{L^5} = \frac{M}{L^3 T} \Rightarrow [D] = \frac{L^2}{T}$$

$$u(x,t) = u(x,t,u_0,D)$$

$$[u] = [x^a \cdot t^b \cdot (u_0)^c \cdot D^d]$$

$$[u] = ML^{-3} = L^a \cdot T^b \cdot \left(\frac{M}{L^3}\right)^c \cdot \left(\frac{L^2}{T}\right)^d$$

$$\begin{cases} 1 = c \\ -3 = a - 3c + 2d \\ 0 = b - d \end{cases} \Rightarrow \quad -3 = a - 3 + 2d \qquad b = d \qquad a + 2d = 0.$$

a-free

$b = d = -\frac{a}{2}$

$$\Rightarrow u(x,t) = \alpha u_0 \cdot \left(\frac{x}{\sqrt{Dt}}\right)^a$$

$$= u_0 F\Big(\underbrace{\frac{x}{\sqrt{Dt}}}_{\eta}\Big) = u_0 F(\eta), \quad \eta = \frac{x}{\sqrt{Dt}}$$

$$\frac{\partial u}{\partial t} = u_0 \cdot F'(\eta)\cdot\frac{\partial \eta}{\partial t} = u_0 F'(\eta)\cdot\left(\frac{-x}{2\sqrt{Dt^3}}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = u_0 F''(\eta)\cdot\frac{1}{Dt}$$

$\Rightarrow$

$$D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow$$

$$\cancel{\frac{D u_0}{Dt}} \cdot F'' = u_0 F' \cdot \left(\frac{-x}{2D^{\frac{1}{2}} t^{3/2}}\right)$$

$$F'' = -\frac{1}{2}\underbrace{\frac{x}{D^{1/2} t^{1/2}}} \cdot F' = -\frac{1}{2}\eta \cdot F'$$

$$\begin{cases} F'' = -\frac{1}{2}\eta F', \quad 0<\eta<\infty \\ F(0) = 1 \\ F(\infty) = 0 \end{cases}$$

$u(x,t) = u_0 F(\eta)$

$\eta = 0 \Leftrightarrow x = 0$

$F(0) = 1$

$\eta \to \infty \Leftrightarrow x \to \infty$

$$G = F' \Rightarrow G' = -\frac{1}{2}\eta G$$

$$G = e^{-\eta^2/4}$$

$$F(\eta) = \beta + \alpha\int_0^\eta e^{-s^2/4}\,ds$$

$$F(0) = \beta = 1$$

$$F(\infty) = 1 + \alpha\underbrace{\int_0^\infty e^{-s^2/4}\,ds}_{\sqrt{\pi}} = 0 \Rightarrow \alpha = -\frac{1}{\sqrt{\pi}}$$

$$\Rightarrow F(\eta) = 1 - \frac{1}{\sqrt{\pi}}\int_0^\eta e^{-s^2/4}\,ds, \quad \operatorname{erfc}(\eta) = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-r^2}\,dr$$

$$\left[\, u(x,t) = u_0 \cdot \operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}\right)\right]$$ ← solution of this BVP.

Now let's find a fundamental solution for $u_t - \Delta u = 0$ in any dimension.

Assume $u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$

Dilation scaling: $u \longmapsto \lambda^\alpha u(\lambda^\beta x, \lambda t)$

Let $\lambda = \frac{1}{t}$ $\quad u(x,t) = \frac{1}{t^\alpha} v\big(\underbrace{\tfrac{x}{t^\beta}}_{y}\big)$

$$v(y) := u(y,1)$$

Plug this int $u_t - \Delta u = 0$:

$$-\alpha t^{-(\alpha+1)} v(y) - \beta t^{-(\beta+1)} x t^{-\alpha} Dv(y) - \frac{1}{t^{\alpha+2\beta}} \Delta v(y) = 0$$

$$\Rightarrow \alpha t^{-(\alpha+1)} v(y) + \beta \cdot t^{-(\alpha+1)} \cdot y \, Dv(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0$$

since $\frac{x}{t^\beta} = y$

To match powers of t, we take $\alpha + 1 = \alpha + 2\beta$

$$\Rightarrow \boxed{\beta = \tfrac{1}{2}}$$

$$\Rightarrow \alpha v + \tfrac{1}{2} y \cdot Dv + \Delta v = 0$$

Let $v(y) = w(|y|)$ - radial form of solution

$r = |y|$ $\qquad v(y) = w(r)$

Differentiate wrt r: $\left[\alpha = \frac{n}{2}\right]$

$$(r^{n-1} w')' + \tfrac{1}{2}(r^n w)' = 0$$

$$\Rightarrow r^{n-1} w' + \tfrac{1}{2} r^n w = A \leftarrow \text{const indep. of } r$$

$$w' = -\tfrac{1}{2} r w \qquad \text{If } \left.\begin{array}{l} \lim_{r\to\infty} w = 0 \\ \lim_{r\to\infty} w' = 0 \end{array}\right\} \Rightarrow A = 0.$$

$$w = B e^{-r^2/4}$$

$$\Rightarrow u = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) = \underbrace{\frac{B}{t^{n/2}} \cdot e^{-|x|^2/4t}}_{\text{this function solves the heat eqn in } \mathbb{R}^n.}$$

$\alpha = \frac{n}{2} \quad \beta = \frac{1}{2}$

<u>Def.</u> $$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t > 0 \quad x \in \mathbb{R}^n \\ 0, & t < 0, \quad x \in \mathbb{R}^n \end{cases}$$

Fundam. solution of heat eqn.

$\Phi(x,t)$ dep. only on $|x|$ $\Rightarrow$ radial solution.

Lemma. $\int_{\mathbb{R}^n} \Phi(x,t)\,dx = 1$

$$\int_{\mathbb{R}^n} \Phi(x,t)\,dx = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x|^2/4t}\,dx =$$

$$= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|x|^2}\,dz = \frac{1}{\pi^{n/2}} \cdot \left(\sqrt{\pi}\right)^n = 1$$

IVP: $\begin{cases} u_t - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$

Solution: $u(x,t) = \cancel{\ }\int_{\mathbb{R}^n} \Phi(x-y,t)\,g(y)\,dy$

$$\left[u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy \right] \quad (*)$$

Thm. $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \Rightarrow$ for u given by $(*)$

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$

(ii) $u_t(x,t) = \Delta u(x,t)$, $t > 0$

(iii) $\lim_{\substack{x \to x^0,\, t \to 0 \\ t > 0}} u(x,t) = g(x^0)$, $x^0 \in \mathbb{R}^n$