Math 678.
Lecture 6.

$U \subset \mathbb{R}^n$ open, bounded, $\partial U \in C^1$

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

Let's derive Green's function for this BVP.

I. Take $u \in C^2(U)$ - arbitrary

$\exists x \in U$, $\varepsilon > 0$ $B(x, \varepsilon) \subset U$

We'll use Green's formula:

$$\int_U (u \Delta v - v \Delta u)\, dx = \int_{\partial U} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu}\right) dS$$

Take $V_\varepsilon := U \setminus B(x, \varepsilon)$ and apply $\uparrow$ to $u = u(y)$, $v = \Phi(y-x)$

$$\int_{V_\varepsilon} (u(y) \underbrace{\Delta \Phi(y-x)}_{0} - \Phi(y-x) \Delta u)\, dy = \int_{\partial V_\varepsilon} \left(u(y) \frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\right) dS(y)$$

$\partial V_\varepsilon = \partial U \cup \partial B(x, \varepsilon)$

$$\left| \int_{\partial B(x,\varepsilon)} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS \right| \leq C \varepsilon^{n-1} \max_{\partial B(x,\varepsilon)} |\Phi|$$

$$\int_{\partial B(x,\varepsilon)} u(y) \underbrace{\frac{\partial \Phi}{\partial \nu}(y-x)}_{\frac{1}{n\alpha(n)\varepsilon^{n-1}}} dS = \fint_{\partial B(x,\varepsilon)} u(y)\, dS(y) \longrightarrow u(x)$$

$$\varepsilon \to 0 \qquad -\int_{V_\varepsilon = U} \Phi(y-x) \Delta u\, dy = u(x) + \int_{\partial U} u(y) \frac{\partial \Phi}{\partial \nu}(y-x)\, dS - \int_{\partial U} \Phi(y-x) \frac{\partial u}{\partial \nu}(y)\, dS$$

$$\Rightarrow u(x) = \int_{\partial V} \Phi(y-x)\frac{\partial u}{\partial \nu}\,dS - \int_{\partial V} u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\,dS - \int_{V} \Phi(y-x)\,\Delta u\,dy$$

Issue: $\frac{\partial u}{\partial \nu}$ is unknown.

Need to compute a corrector function:

$$\varphi^x = \varphi^x(y) \text{ s.t. } \begin{cases} \Delta \varphi^x = 0 & \text{in } V \\ \varphi^x = \Phi(y-x) & \text{on } \partial V \end{cases}$$

$$\underbrace{G(x,y) := \Phi(y-x) - \varphi^x(y)}_{\text{Green's function for region } V}, \quad x, y \in V,\ x \neq y.$$

This $G(x,y)$ will give representation formula w/o computing $\frac{\partial u}{\partial \nu}$ term: [write $\Phi(y-x) = G + \varphi^x$]

$$u(x) = \int_{\partial V} \Phi(y-x)\frac{\partial u}{\partial \nu}\,dS - \int_{\partial V} u(y)\frac{\partial G}{\partial \nu}\,dS - \int_{\partial V} u(y)\frac{\partial \varphi^x}{\partial \nu}\,dS - \int_{V} G(x,y)\,\Delta u\,dy - \int_{V} \varphi^x(y)\,\Delta u\,dy$$

Green's formula for

$$-\int_{V} \varphi^x(y)\,\Delta u(y)\,dy = \int_{\partial V}\left(u(y)\frac{\partial \varphi^x}{\partial \nu} - \varphi^x \cdot \frac{\partial u}{\partial \nu}\right)dS$$

$$= \int_{\partial V} u(y)\frac{\partial \varphi^x}{\partial \nu}\,dy - \int_{\partial V} \Phi(y-x)\frac{\partial u}{\partial \nu}\,dS$$

After canceling like terms:

$$u(x) = -\int_{\partial V} u(y)\underbrace{\frac{\partial G}{\partial \nu}}_{D_y G(x,y)\cdot \nu(y)}\,dS - \int_{V} G(x,y)\,\Delta u\,dy$$

II. Now suppose $\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$

$u \in C^2(\bar{U})$

$$\Rightarrow \boxed{u(x) = -\int_{\partial U} g(y) \frac{\partial G}{\partial \nu} dS + \int_U G(x,y) f(y)\, dy}$$

Representation formula for U.

<u>Lemma</u>: $G(x,y) = G(y,x)$ for $x \neq y$, $x, y \in U$

<u>Half-space</u>

$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid (x_1, \dots x_{n-1}, x_n) = x,\ x_n > 0\}$

$\tilde{x} = (x_1, \dots x_{n-1}, -x_n)$ - reflection of x in the plane $\partial \mathbb{R}^n_+$.

Take $\varphi^x(y) := \cancel{\Phi(y-x)}\ \Phi(\tilde{y} - \tilde{x})$

$\Phi(\tilde{y} - \tilde{x}) = (y_1 - x_1, \dots, -y_n + x_n)$

Notice $\varphi^x(y) = \Phi(y-x)$, $y \in \partial \mathbb{R}^n_+$

$\Phi(y-x) = (y_1 - x_1, \dots\ y_n - x_n)$

$$\Rightarrow \begin{cases} \Delta \varphi^x = 0 & \text{in } \mathbb{R}^n_+ \\ \varphi^x = \Phi(y-x) & \text{on } \partial \mathbb{R}^n_+ \end{cases}$$

$G(x,y) := \Phi(y-x) - \Phi(\tilde{y} - \tilde{x})$, $x, y \in \mathbb{R}^n_+$

$$\frac{\partial G}{\partial y_n}(x,y) = \frac{\partial \Phi}{\partial y_n}(y-x) + \frac{\partial \Phi}{\partial y_n}(\tilde{y} - x) =$$

$$= \frac{-1}{n\alpha(n)} \left(\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-x|^n} \right)$$

for $y \in \partial \mathbb{R}^n_+$ $y_n = 0 \Rightarrow \frac{\partial G}{\partial y_n}(x,y) = \frac{2x_n}{n\alpha(n)} \cdot \frac{1}{|y-x|^n}$

$$\frac{\partial G}{\partial \nu}(x,y) = -\frac{\partial G}{\partial y_n}(x,y)$$

Suppose $\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$

We are going to check that

$$\underbrace{u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\,dy}_{\text{Poisson formula}} \quad \text{solves this BVP}$$

$$K(x,y) = \frac{2x_n}{n\alpha(n)|x-y|^n} \text{ — Poisson kernel}$$

I. $K(x,y) = -\frac{\partial G}{\partial y_n}(x,y)$

Since $y \longmapsto G(x,y)$ harmonic for $x \neq y$

and $G(x,y) = G(y,x)$

$\Rightarrow x \longmapsto G(x,y)$ harmonic for $x \neq y$

$\Rightarrow x \longmapsto -\frac{\partial G}{\partial y_n}(x,y) = K(x,y)$ — also harmonic $\forall x \in \mathbb{R}^n_+,\ y \in \partial\mathbb{R}^n_+$

This means that

$$\Delta u(x) = \int_{\partial\mathbb{R}^n_+} \underbrace{\Delta_x K(x,y)}_{=0} g(y)\,dy = 0$$

$u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ because $K(x,y)$ is smooth.

II. To show:

$$\lim_{\substack{x \to x^0 \\ x \in \mathbb{R}^n_+}} u(x) = g(x^0), \quad \forall x^0 \in \partial\mathbb{R}^n_+$$

Let $x^0 \in \partial\mathbb{R}^n_+$, $\varepsilon > 0$

Fix $\delta > 0$ s.t. $|g(y) - g(x^0)| < \varepsilon$ if $|y - x^0| < \delta$, $y \in \partial\mathbb{R}^n_+$

if $|x-x^0|<\frac{\delta}{2}$, $x \in \mathbb{R}^n_+ \Rightarrow$ since $\int\limits_{\partial\mathbb{R}^n_+} \underset{\overset{\vee}{0}}{K(x,y)} = 1$

$$|u(x)-g(x^0)| =$$

$$= \left| \int\limits_{\partial\mathbb{R}^n_+} K(x,y)\big(g(y)-g(x^0)\big)\,dy \right| \leq$$

$$\leq \underbrace{\int\limits_{\partial\mathbb{R}^n_+ \cap B(x^0,\delta)} K(x,y)\,|g(y)-g(x^0)|\,dy}_{\overset{\wedge}{\varepsilon}} + \int\limits_{\partial\mathbb{R}^n_+ \setminus B(x^0,\delta)} K(x,y)\,|g(y)-g(x^0)|\,dy$$

since $|g(y)-g(x^0)|<\varepsilon$ here