Math 678.
Lecture 4.

MVT for Laplace eqn

$u \in C^2(U)$, harmonic $\Rightarrow$

$$u(x) = \fint_{\partial B(x,r)} u\, dS = \fint_{B(x,r)} u\, dy, \quad \forall B(x,r) \subset U$$

Converse to MVT:

If $u \in C^2(U)$ satisfies $u(x) = \fint_{\partial B(x,r)} u\, dS$ for any $B(x,r) \subset U \Rightarrow u$ - harmonic.

Pf: $\Delta u \neq 0$ in some ball $B(x,r) \subset U$.

Then $\varphi(r) = \fint_{\partial B(x,r)} u\, dS = u(x)$

$\varphi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\, dy$ from last Theorem (MVT).

Since $\varphi(r) = u(x) \Rightarrow 0 = \varphi'(r) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)\, dy \neq 0$

$\Rightarrow \Delta u = 0$ contradiction.

Maximum Principle:

$U \subset \mathbb{R}^n$ - open bounded

Thm. SMP

$u \in C^2(U) \cap C(\bar{U})$, harmonic in U

$\Rightarrow$ 1) $\max_{\bar{U}} u = \max_{\partial U} u$

2) if U - connected, $u(x_0) = \max_{\bar{U}} u$, $x_0 \in U$

$\Rightarrow u \equiv \text{const}$ in U.

<u>Pf.</u> If $\exists x_0 \in U$ s.t. $u(x_0) = \max_{U} u = M$

Fix r s.t. $0 < r < dist(x_0, \partial U)$

$\Rightarrow$ apply MVT in $B(x_0, r)$:

$$M = u(x_0) = \fint_{B(x_0,r)} u \, dy \leq M$$

$=$ holds only in $u \equiv M$ on $B(x_0, r)$.

Filling entire domain will balls, we get $u \equiv M$ on U if U is connected.

<u>Corollary</u>: U-connected

$u \in C^2(U) \cap C(\bar{U})$

If u satisfies $\begin{cases} \Delta u = 0 , & u \in U \\ u = g , & u \in \partial U , \quad g \geq 0 \end{cases}$

If $g > 0$ at least at one point of ∂U, then $u > 0$ everywhere on U.

<u>Uniqueness Thm</u>:

$$\begin{cases} -\Delta u = f , & \text{on } U \qquad f \in C(U) \\ u = g , & \text{on } \partial U \qquad g \in C(\partial U) \end{cases}$$

This IVP has at most one sol. $u \in C^2(U) \cap C(\bar{U})$

<u>Pf</u>: Suppose u_1, u_2 - 2 sol'ns

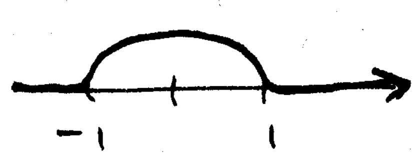
$$W := u_1 - u_2 \Rightarrow \begin{cases} \Delta W = 0 , & U \\ W = 0 , & \partial U \end{cases}$$

$$\left.\begin{aligned} 0 &= \max_{\partial U} W = \max_{U} W \\ 0 &= \max_{\partial U}(-W) = \max_{U}(-W) \end{aligned}\right\} \Rightarrow W \equiv 0 \text{ on } U.$$

Regularity:

Thm: $u \in C(U)$, satisfies MVT for all balls $B(x,r) \subset U \Rightarrow u \in C^\infty(U)$

Pf: $\eta(x) = \begin{cases} C e^{\frac{1}{|x|^2-1}}, & |x|<1 \\ 0, & |x| \geq 1 \end{cases}$



$\eta_\varepsilon := \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$

$\eta \in C^\infty(\mathbb{R}^n)$, $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \eta(x) = 1 \leftarrow$ by choice of C

$\Rightarrow \int_{\mathbb{R}^n} \eta_\varepsilon \, dx = 1$, $\operatorname{supp}(\eta_\varepsilon) \subset B(0,\varepsilon)$

Mollification of u: $u_\varepsilon := \eta_\varepsilon * u$

in $U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}$

$$u_\varepsilon(x) = \int_U \eta_\varepsilon(x-y)u(y)\,dy = \int_{B(0,\varepsilon)} \eta_\varepsilon(y)\,u(x-y)\,dy$$

$$D^\alpha u_\varepsilon(x) = \int_U D^\alpha \eta_\varepsilon(x-y)\,u(y)\,dy \in C \Rightarrow u_\varepsilon \in C^\infty(U).$$

To show: $u \equiv u^\varepsilon$ on U_ε

Fix $x \in U_\varepsilon \Rightarrow$

$$u^\varepsilon(x) = \int_U \eta_\varepsilon(x-y)u(y)\,dy = \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} \eta\left(\frac{|x-y|}{\varepsilon}\right)u(y)\,dy$$

$$= \frac{1}{\varepsilon^n}\int_0^\varepsilon \int_{\partial B(x,r)} \eta\left(\frac{|x-y|}{\varepsilon}\right)u(y)\,dS(y)\,dr =$$

$$= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)\left(\int_{\partial B(x,r)} u(y)\,dS(y)\right)dr = \text{by MVT}$$

$$= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right)\cdot n\alpha(n)\cdot r^{n-1}\cdot u(x)\,dr =$$

$$= n\alpha(n)\cdot\left(\int_0^{\varepsilon} \eta_\varepsilon(r)\cdot r^{n-1}\,dr\right)u(x) =$$

$$= u(x)\cdot\underbrace{\int_0^{\varepsilon}\int_{\partial B(0,\varepsilon)} \eta_\varepsilon(r)\,dS\,dr}_{\int_{B(0,\varepsilon)}} = u(x)\cdot\underbrace{\int_{B(0,\varepsilon)} \eta_\varepsilon(y)\,dy}_{1} = u(x)$$

$$\Rightarrow\ u \equiv u^\varepsilon \text{ on } U_\varepsilon \ \Rightarrow\ u\in C^\infty(U_\varepsilon)\ \forall\varepsilon>0.$$

$$\Rightarrow\ u\in C^\infty(U)$$