Math 678

Lecture 3.

Thm (cont.)

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\,dy, \quad \Phi(x) := \begin{cases} -\frac{1}{2\pi}\log|x|, & n=2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

(i) $u \in C^2(\mathbb{R}^n) \leftarrow$ done

(ii) $-\Delta u = f$ in $\mathbb{R}^n$

$$\Delta u(x) = \underbrace{\int_{B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\,dy}_{I_\varepsilon} + \underbrace{\int_{\mathbb{R}^n - B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\,dy}_{J_\varepsilon}$$

$$|I_\varepsilon| = \left| \int_{B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\,dy \right| \leq C \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} |\Phi(y)|\,dy$$

Case $n=2$: $\displaystyle\int_{B(0,\varepsilon)} |\Phi(y)|\,dy = \frac{1}{2\pi} \int_{B(0,\varepsilon)} \log|y|\,dy = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\varepsilon r \log|r|\,dr$

u $dv = r\,dr$

$du = \frac{1}{r}\,dr$ $\frac{r^2}{2}$

$$\int_0^\varepsilon r \log r\,dr = \lim_{\delta \to 0} \left(\frac{r^2}{2} \log r\right)\Big|_\delta^\varepsilon - \int_0^\varepsilon \frac{r}{2}\,dr$$

$$= \frac{\varepsilon^2 \log \varepsilon}{2} - \frac{\varepsilon^2}{4}$$

$$\Rightarrow \int_{B(0,\varepsilon)} |\Phi(y)|\,dy \leq \frac{\varepsilon^2 \log \varepsilon}{2} \Rightarrow |I_\varepsilon| \leq \boxed{C \varepsilon^2 \log \varepsilon}$$

Case $n \geq 3$: $\displaystyle \Phi(x) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}}$

$$\int_{B(0,\varepsilon)} |\Phi(y)|\,dy = \frac{1}{n(n-2)\alpha(n)} \int_{B(0,\varepsilon)} \frac{1}{|y|^{n-2}}\,dy =$$

$$= \frac{1}{n(n-2)\alpha(n)} \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{|y|^{n-2}}\,dS(y)\,dr =$$

$$= \frac{1}{n(n-2)\alpha(n)} \int_0^{\varepsilon} \frac{1}{r^{n-2}} \cdot \Big(\underbrace{\int_{\partial B(0,r)} dS(y)}_{n\alpha(n)\cdot r^{n-1}} \Big) dr =$$

$n\alpha(n)\cdot r^{n-1}$ – surface area of a sphere

$$= \frac{1}{n-2} \int_0^{\varepsilon} r\, dr = \frac{1}{n-2} \cdot \frac{\varepsilon^2}{2}$$

$$\Longrightarrow \int_{B(0,\varepsilon)} |\Phi(y)|\, dy \le C \cdot \varepsilon^2 \quad \text{for all } n \geq 3.$$

$$\boxed{\text{So } \lim_{\varepsilon \to 0} |I_\varepsilon| = 0}$$

Now take $J_\varepsilon = \int_{\mathbb{R}^n - B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y)\, dy$

Integration by parts:

$$\int_U u_{x_i} v\, dx = -\int_U u v_{x_i}\, dx + \int_{\partial U} u v \cdot \nu^i\, ds$$

ν – outward normal

$$J_\varepsilon = -\int_{\mathbb{R}^n - B(0,\varepsilon)} D\Phi \cdot Df(x-y) + \int_{\partial B(0,\varepsilon)} \Phi(y) \underbrace{Df(x-y)\cdot \nu}_{\frac{\partial f}{\partial \nu}(x-y) \text{ by defn}}\, dS(y)$$

$$= \underbrace{-\int_{\mathbb{R}^n - B(0,\varepsilon)} D\Phi \cdot Df(x-y)}_{K_\varepsilon} + \underbrace{\int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y)\, dS(y)}_{L_\varepsilon}$$

$$|L_\varepsilon| \le \|Df\|_{L^\infty(\mathbb{R}^n)} \cdot \int_{\partial B(0,\varepsilon)} |\Phi(y)|\, dS(y)$$

$n=2$ $\quad \int_{\partial B(0,\varepsilon)} \log \varepsilon\, dS(y) = 2\pi\varepsilon \cdot \log \varepsilon \xrightarrow[\varepsilon\to 0]{} 0$

$\Phi(y) \sim \log|y|$

$n \geq 3$

$\cancel{\Phi(y) \sim \frac{1}{|x|^{n-2}} \quad \frac{1}{n(n-2)\alpha(n)} \int_{\partial B(0,\varepsilon)} \frac{1}{\varepsilon^{n-2}}\, dS(y)} = $

$n\geq 3$ $\quad \varphi(y) = \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}}$

$$\Rightarrow \int_{\partial B(0,\varepsilon)} |\varphi(y)|\, dS(y) \leq \frac{1}{\cancel{n}(n-2)\cancel{\alpha(n)}\cdot\varepsilon^{n-2}}\cdot \cancel{n\alpha(n)}\,\varepsilon^{n-1} = \frac{\varepsilon}{n-2}$$

$\Rightarrow |L_\varepsilon| \leq C\cdot\varepsilon$ when $n\geq 3$

as $\varepsilon\to 0$ $\downarrow 0$

Now finally take K_ε:

$$K_\varepsilon = +\int_{\mathbb{R}^n - B(0,\varepsilon)} D\varphi(y)\cdot Df(x-y)\,dy \underset{\text{by parts}}{=}$$

$$= -\int_{\mathbb{R}^n - B(0,\varepsilon)} \Delta\varphi(y)\cdot f(x-y)\,dy + \underbrace{\int_{\partial B(0,\varepsilon)} D\varphi(y)\, f(x-y)\cdot\nu\, dS(y)} =$$

$\Delta\varphi = 0$ since

φ- harmonic on $\mathbb{R}^n\setminus B(0,\varepsilon)$

$$= \int_{\partial B(0,\varepsilon)} \frac{\partial\varphi}{\partial\nu}(y)\, f(x-y)\, dS(y)$$

$$D\varphi(y) = -\frac{1}{n\alpha(n)}\frac{y}{|y|^n}, \quad y\neq 0$$

$$\nu = -\frac{y}{|y|} = -\frac{y}{\varepsilon} \text{ on } \partial B(0,\varepsilon)$$

$$\frac{\partial\varphi}{\partial\nu}(y) = D\varphi(y)\cdot\nu = \frac{|y|^2}{n\alpha(n)\cdot|y|^n\cdot\varepsilon} = \frac{1}{n\alpha(n)\,|y|^{n-1}}$$

$$|y|=\varepsilon \quad = \frac{1}{n\alpha(n)\,\varepsilon^{n-1}}$$

$$\Rightarrow K_\varepsilon = \frac{1}{n\alpha(n)\varepsilon^{n-1}}\int_{\partial B(0,\varepsilon)} f(x-y)\, dS(y) =$$

$$= \frac{1}{n\alpha(n)\varepsilon^{n-1}}\int_{\partial B(x,\varepsilon)} f(y)\, dS(y) = \fint_{\partial B(x,\varepsilon)} f(y)\, dS(y) = f(x)$$

$$\boxed{\Delta u = -f}$$

Some formulas we used:

Integration by parts:

$$\int_U u_{x_i} v \, dx = \int_{\partial U} u v \, \nu^i \, dS - \int_U u v_{x_i} \, dx$$

Directional derivative:

$$Df \cdot \nu = \frac{\partial f}{\partial \nu}, \quad \nu \text{ - outward normal}$$

Volume of a ball in $\mathbb{R}^n$ of radius r: $\alpha(n) \cdot r^n$

where $\alpha(n) = \frac{\pi}{\Gamma(\frac{n}{2}+1)}$ = volume of a unit ball in $\mathbb{R}^n$

Surface area of a sphere in $\mathbb{R}^n$: $n\alpha(n) \cdot r^{n-1}$

Averages:

$$\fint_{B(x,r)} f \, dy = \frac{1}{\alpha(n) \cdot r^n} \cdot \int_{B(x,r)} f \, dy$$

$$\fint_{\partial B(x,r)} f \, dy = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} f \, dy$$

We fixed the faulty calculation we had before, we can call $-\Delta \Phi = \delta_0$ in $\mathbb{R}^n$

$$-\Delta u(x) = -\int \Delta_x \Phi(x-y) f(y) \, dy = \int \delta_0(x-y) f(y) \, dy = f(x)$$

$$\cancel{\delta_0(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}}$$

$$\int \delta_0(x) \, dx = 1$$

$$\int \delta_a(x) f(x) \, dx = f(a)$$

Mean value formulas.

$U \subset \mathbb{R}^n$ open

$u(x)$ – harmonic in U $(\Delta u = 0)$

Thm: (MVT for Laplace equation)

$u \in C^2(U)$ harmonic $\Rightarrow$

$$u(x) = \fint_{\partial B(x,r)} u\,dS = \fint_{B(x,r)} u\,dy \quad \text{for any ball } B(x,r) \subset U$$

Pf:

$$\varphi(r) := \fint_{\partial B(x,r)} u(y)\,dS(y) = \fint_{\partial B(0,1)} u(x+rz)\,dS(z) \qquad \begin{aligned} y &= x + rz \\ z &= \frac{y-x}{r} \end{aligned}$$

$$\varphi'(r) = \fint_{\partial B(0,1)} Du(x+rz)\cdot z\,dS(z) = \fint_{\partial B(x,r)} Du(y)\cdot \underbrace{\frac{y-x}{r}}_{\nu}\,dS(y)$$

$$= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\,dS(y) = \frac{1}{\underbrace{n\alpha(n)r^{n-1}}_{\text{surface area}}} \cdot \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\,dS(y) =$$

$$\left[\text{Green's Thm: } \int_{\partial U} \frac{\partial u}{\partial \nu}\,dS = \int_U \Delta u\,dx\right]$$

$$= \frac{1}{n\alpha(n)\cdot r^{n-1}} \cdot \int_{B(x,r)} \Delta u\,dy = 0 \quad \text{since } u \text{ – harmonic in } U$$

$\Rightarrow \varphi'(r) = 0. \Rightarrow \varphi \equiv \text{const on } U$

$$\varphi(r) = \varphi(0) = \lim_{r\to 0} \fint_{\partial B(x,r)} u(y)\,dS(y) = u(x)$$

$$\varphi(r) = \fint_{\partial B(x,r)} u(y)\,dS(y)$$

By polar coordinates:

$$\int_{B(x,r)} u\,dy = \int_0^r \iint_{\partial B(x,s)} u\,dS\,ds = u(x)\int_0^r n\,\alpha(n)\cdot s^{n-1}\,ds$$

$$= \cancel{n}\,\alpha(n)\cdot u(x)\int_0^r n\,s^{n-1}\,ds = \alpha(n)\cdot u(x)\cdot r^n$$

$$\Rightarrow u(x) = \frac{\int_{B(x,r)} u\,dy}{\underbrace{\alpha(n)\cdot r^n}_{\text{volume of } B(x,r)}} = \fint_{B(x,r)} u\,dy$$